On the $KU_{(p)}$-local stable homotopy category

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Based on joint work in progress with Francis Clarke & Martin Crossley.

§1. The ring A

Notation

Throughout p will be an odd prime.

We fix a choice of q primitive mod p^2.

We write K for $KU_{(p)}$, the p-local complex K-theory spectrum.

Define polynomials $\Theta_n(X) \in \mathbb{Z}_{(p)}[X]$, for each integer $n \geqslant 0$, by

$$\Theta_n(X) = \prod_{i=1}^{n} (X - q_i),$$

where $q_i = q^{(-1)^i \lfloor i/2 \rfloor}$.

For example,
$\Theta_4(X) = (X-1)(X-q)(X-q^{-1})(X-q^2)$.

We have a decreasing filtration

$$\mathbb{Z}_{(p)}[X] \supset \ldots \supset (\Theta_n(X)) \supset (\Theta_{n+1}(X)) \supset \ldots$$

Definition

We define the topological ring A to be the completion of $\mathbb{Z}_{(p)}[X]$ with respect to the filtration by the ideals $(\Theta_n(X))$,

$$A = \lim_{\leftarrow} \frac{\mathbb{Z}_{(p)}[X]}{(\Theta_n(X))}.$$

Theorem

There is an isomorphism of rings $A \cong K^0(K)$. The variable X corresponds to the Adams operation Ψ^q.

The elements of A can be expressed uniquely as infinite sums

$$\sum_{n \geqslant 0} a_n \Theta_n(\Psi^q),$$

where $a_n \in \mathbb{Z}_{(p)}$.

Since Ψ^q acts on $\pi_{2i}(K)$ as multiplication by q^i, the action of $\Theta_m(\Psi^q)$ on $\pi_{2i}(K)$ is zero for $-(m/2) < i < (m+1)/2$.

For each $m \geqslant 0$ we define

$$A_m = \left\{ \sum_{n \geqslant m} a_n \Theta_n(\Psi^q) : a_n \in \mathbb{Z}_{(p)} \right\}.$$

This is an ideal in A, consisting of the operations which are zero on the coefficient groups $\pi_{2i}(K)$ for $-(m/2) < i < (m+1)/2$.

Theorem

- The element $\sum_{n \geqslant 0} a_n \Theta_n(\Psi^q)$ is a unit in A if and only if $\sum_{n=0}^{i-1} a_n \Theta_n(q_i)$ is a p-local unit for $i = 1, \dots, p-1$.
- A is not Noetherian; for each $m \geqslant 1$, the ideal A_m is not finitely generated.
- For all $m \geqslant 0$, there is an isomorphism of $\mathbb{Z}_{(p)}$-modules,

$$A_m/\Theta_m(\Psi^q)A \cong \bigoplus_{i=1}^{m} \mathbb{Z}_p/\mathbb{Z}_{(p)}.$$

§2. Discrete A-modules

Definition

A *discrete A-module* M is an A-module such that the action map $A \times M \to M$ is continuous with respect to the discrete topology on M and the resulting product topology on $A \times M$.

Equivalently, for each $x \in M$, there is some n such that $A_n x = 0$.

Example

For any spectrum X, $K_0(X)$ is a discrete A-module:

$$A \otimes K_0(X) \to A \otimes K_0(K) \otimes K_0(X) \to K_0(X),$$

where the first map arises from the coaction and the second comes from the Kronecker pairing.

Definition

An A-module M is *locally finitely generated* if, for every $x \in M$, the submodule Ax of M is finitely generated over $\mathbb{Z}_{(p)}$.

Theorem

An A-module is discrete if and only if it is locally finitely generated.

Sketch Proof

If M is discrete and $x \in M$, then for some n, $Ax = (A/A_n)x$ and A/A_n has rank n over $\mathbb{Z}_{(p)}$.

Now suppose M is l.f.g. and let $x \in M$.

Step 1. There is some n such that $\Theta_n(\Psi^q)x = 0$.

- M free and finite rank over $\mathbb{Z}_{(p)}$: use that M is a *slender* group.
- M torsion and finitely generated over $\mathbb{Z}_{(p)}$, $p^s M = 0$. A key ingredient is that one can find $f \in \mathbb{Z}_{(p)}[\Psi^q]$ and $i \in \mathbb{Z}$ such that $fM = 0$ such that $f(q_i) \equiv 0 \bmod p$.

Step 2. If $\Theta_n(\Psi^q)x = 0$ then $A_n x = 0$.

- The action map $A \to Ax$, $a \mapsto ax$, factors via $A/\Theta_n(\Psi^q)A$.
- Consider the restriction $A_n/\Theta_n(\Psi^q)A \cong \oplus_{i=1}^n \mathbb{Z}_p/\mathbb{Z}_{(p)} \to Ax$. This is a homomorphism from a divisible group to a group with no non-zero infinitely p-divisible element and so must be zero. $\square$

Definition

We define the *category of discrete* A*-modules*, $\mathcal{C}A$, to be the full subcategory of the category of A-modules whose objects are discrete A-modules.

It is easy to see that $\mathcal{C}A$ is a cocomplete abelian category.

§3. Bousfield's category of K-theory modules

A. K. Bousfield, On the homotopy theory of K-local spectra at an odd prime, Amer. J. Math. 107 (1985), 895–932.

[A. K. Bousfield, A classification of K-local spectra, J. Pure Appl. Algebra 66 (1990), 121–163.

J. J. Wolbert, Classifying modules over K-theory spectra, J. Pure Appl. Algebra 124 (1998), 289–323.]

Let $R = \mathbb{Z}_{(p)}[\mathbb{Z}_{(p)}^\times]$ (where $j \in \mathbb{Z}_{(p)}^\times$ corresponds to Ψ^j).

Bousfield's category $\mathcal{A}(p)$ is the full subcategory of the category of R-modules whose objects M satisfy the following conditions:

(i) If M is finitely generated over $\mathbb{Z}_{(p)}$, then

(a) for each $j \in \mathbb{Z}_{(p)}^\times$, Ψ^j acts on $M \otimes \mathbb{Q}$ by a diagonalizable matrix whose eigenvalues are integer powers of j,

(b) for each $m \geqslant 1$, the action of $\mathbb{Z}_{(p)}^\times$ on $M/p^m M$ factors through the quotient homomorphism $\mathbb{Z}_{(p)}^\times \to (\mathbb{Z}/p^k)^\times$ for sufficiently large k.

(ii) For each $x \in M$, the submodule $Rx \subset M$ is finitely generated over $\mathbb{Z}_{(p)}$ and satisfies condition (i).

Theorem

Bousfield's category $\mathcal{A}(p)$ is isomorphic to the category $\mathcal{C}A$ of discrete A-modules.

Sketch Proof

We have $R \subset A$ and we have a formula for writing Ψ^j as an element of A.

Assume M is finitely generated over $\mathbb{Z}_{(p)}$.

If M is a Bousfield module, the R-action extends uniquely to an A-action, making M a discrete A-module:

- By (ia) there is some r such that $\Theta_r(\Psi^q)$ acts as zero on $M \otimes \mathbb{Q}$.
- Given e, using (ib) one can show there is some s such that $\Theta_s(\Psi^q)M \equiv 0 \bmod p^e$.
- Deduce that there is some m such that $\Theta_m(\Psi^q)M = 0$.
- Let the infinite sum $\sum_{n \geqslant 0} a_n \Theta_n(\Psi^q)$ act as the finite sum $\sum_{n=0}^{m-1} a_n \Theta_n(\Psi^q)$.

Let M be an A-module that is finitely generated over $\mathbb{Z}_{(p)}$. Then M is a Bousfield module:

- Rational diagonalizability.
  - Since $\Theta_m(\Psi^q)M = 0$, the matrix of the action of Ψ^q on $M \otimes \mathbb{Q}$ has minimal polynomial dividing $\Theta_m(X)$. So it has distinct rational roots, which are integer powers of q, and the matrix is diagonalizable.
  - For $j \in \mathbb{Z}_{(p)}^{\times}$, we can write $\Psi^j M = \sum_{n=0}^{m-1} F_n(j)\Theta_n(\Psi^q)M = P(\Psi^q)M$. So the eigenvalues of Ψ^j are $P(q_i) = P(q^{(-1)^i\lfloor i/2 \rfloor}) = j^{(-1)^i\lfloor i/2 \rfloor}$ for distinct i.
- p-adic continuity. Write $\Psi^j M = \sum_{n=0}^{m-1} F_n(j)\Theta_n(\Psi^q)M$ and use p-adic continuity of the Laurent polynomials F_n.

□

There is a graded version $\mathcal{A}(p)_*$:

- For $i \in \mathbb{Z}$, there is an automorphism $T^i : \mathcal{A}(p) \to \mathcal{A}(p)$ with $T^i M$ equal to M as a $\mathbb{Z}_{(p)}$-module but with $\Psi^k : T^i M \to T^i M$ equal to $k^i \Psi^k : M \to M$ for each $k \in \mathbb{Z}_{(p)}^\times$.
- An object of $\mathcal{A}(p)_*$ is a collection of objects $M_n \in \mathcal{A}(p)$ for $n \in \mathbb{Z}$ together with isomorphisms $u : TM_n \cong M_{n+2}$ in $\mathcal{A}(p)$ for all n.
- A morphism $f : M \to N$ in $\mathcal{A}(p)_*$ is a collection of morphisms $f_n : M_n \to N_n$ in $\mathcal{A}(p)$ such that $uf_n = f_{n+2}u$ for all n.

Theorem [Bousfield]

The objects of the K-local homotopy category correspond to isomorphism classes of pairs (M, κ), where $M \in \mathcal{A}(p)_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{A}(\mathrm{p})_*}(\mathrm{M}, \mathrm{M})$.

Bousfield also gives information about the morphisms in the category, up to some filtration.

Theorem [Bousfield]

$\mathcal{A}(p)_*$ is equivalent to the category of $K_*(K)$-comodules.

§4. Some constructions in $\mathcal{C}A$

We consider $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A, M)$ as an A-module via $af(b) = f(ab)$ for $a, b \in A$.

Proposition

- The functor U from $\mathbb{Z}_{(p)}$-modules to $\mathcal{C}A$ given by $UM = \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$ is right adjoint to the forgetful functor.
- For any $\mathbb{Z}_{(p)}$-module M, $UM \cong K_0(K) \otimes M$.
- $\mathcal{C}A$ has enough injectives.

Proposition

For any M in $\mathcal{C}A$, there is an exact sequence:

$$0 \to M \xrightarrow{\alpha} UM \xrightarrow{\beta} UM \xrightarrow{\gamma} M \otimes \mathbb{Q} \to 0.$$

The map α is adjoint to the identity map $M \to M$. The map β is given by

$$\beta(f) = \Psi^q \circ f - f \circ \Psi^q,$$

where $f : A \to M$.

This exact sequence leads to:

If $s > 2$, $\mathrm{Ext}^{\mathrm{s}}_{\mathcal{C}\mathrm{A}}(\mathrm{L}, \mathrm{M}) = 0$ for all $L, M \in \mathcal{C}A$.