# Stable and unstable operations in mod p cohomology theories

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## §1. Stable and unstable operations

We will discuss operations between two multiplicative graded cohomology theories $E^*(-)$ and $F^*(-)$ which are commutative and complex orientable.

There are various kinds of operations:

- **Stable operations** $F^*(-) \to E^*(-)$ of degree $l$, given by $E^l(F) = [F, E]^l$.
- **Unstable operations** $F^k(-) \to E^{k+l}(-)$, given by $E^{k+l}(\underline{F}_k)$.
- **Additive operations** $F^k(-) \to E^{k+l}(-)$, given by $PE^{k+l}(\underline{F}_k)$.

The restriction map from stable to unstable operations factors via additive operations, so for each $k \in \mathbb{Z}$ we have maps

$$E^l(F) \to PE^{k+l}(\underline{F}_k) \hookrightarrow E^{k+l}(\underline{F}_k).$$

We can also consider the corresponding objects in the world of co-operations.

- **Stable co-operations**, given by $E_l(F)$, $l \in \mathbb{Z}$.
- **Unstable co-operations**, given by $E_{k+l}(\underline{F}_k)$, $k, l \in \mathbb{Z}$.
- **Additive co-operations**, given by $QE_{k+l}(\underline{F}_k)$, the indecomposables of $E_{k+l}(\underline{F}_k)$.

The stabilization map $\sigma_{k*} : E_{k+l}(\underline{F}_k) \to E_l(F)$, induced by $\sigma_k : \Sigma^\infty \underline{F}_k \to F$, factors via the additive co-operations:

$$E_{k+l}(\underline{F}_k) \to QE_{k+l}(\underline{F}_k) \to E_l(F).$$

Throughout $p$ will be an odd prime. We assume our theories satisfy the following conditions.

- $E^*$ has characteristic $p$.
- The formal group law of $E^*(-)$ has finite height, say $n$.
- The coefficient of the first term in the $p$-series for $E^*(-)$ is invertible.
- The various $E^*(-)$-modules of operations from $F^*(-)$ to $E^*(-)$ are the duals over $E^*$ to the corresponding $E^*$-modules of co-operations.

For example, we can take $E$ to be Morava $K$-theory $K(n)$ and $F$ to be any (multiplicative commutative complex orientable) theory.

## Theorem

- (Bousfield, Kuhn) For each $k, l \in \mathbb{Z}$, there's a map $\Delta^\infty : E^{k+l}(\underline{F}_k) \to E^l(F)$, left-inverse to restriction.
- There is a positive integer $h \leqslant 2\frac{p^n-1}{p-1} + 1$, such that an operation $F^k(-) \to E^{k+l}(-)$ is a component of a stable operation if and only if it is the $h$-fold loop of an operation. Equivalently, a map $\underline{F}_k \to \underline{E}_{k+l}$ is an infinite loop map if and only if it is an $h$-fold loop map.
- For $\rho_k \in E^{k+l}(\underline{F}_k)$, the components of $\Delta^\infty(\rho_k)$ are given by:

  $$(\Delta^\infty(\rho_k))_m = (v_n^E)^{-i}(\Omega^j \rho_k)(v_n^F)^i,$$

  where $i, j \geqslant 0$ are such that $j \geqslant h$ and $m - k = 2(p^n - 1)i - j$.

Let $x^E \in E^2(\mathbb{C}P^\infty)$ be the universal Chern class. Then $E^*(\mathbb{C}P^\infty) = E^*[\![x^E]\!]$,
$E_*(\mathbb{C}P^\infty) = E_*\{\beta_1^E, \beta_2^E, \ldots\}$
where $\beta_i^E$ is dual to $(x^E)^i$.

The $p$-series of $E$ is
$[p]_E(x^E) = x^E +_E x^E +_E \cdots +_E x^E \in E^*[\![x^E]\!]$.
Modulo $p$ this has the form

$$[p]_E(x^E) \equiv {\sum_{i \geqslant 1}}^{E} v_i^E (x^E)^{p^i},$$

where $v_i^E \in E^{-2(p^i-1)}$. The hypotheses on $E$ mean that

$$[p]_E(x^E) \equiv v_n^E (x^E)^{p^n} + \text{higher terms},$$

where $v_n^E \in E^{-2(p^n-1)}$ is invertible.

The universal Chern class $x^F$ represents a map $x^F : \mathbb{C}P^\infty \to \underline{F}_2$. Consider the induced map $x_*^F : E_*(\mathbb{C}P^\infty) \to E_*(\underline{F}_2)$ and write

$$\begin{aligned} b_i &= x_*^F(\beta_i^E) && \in E_*(\underline{F}_2), \\ b(s) &= \textstyle\sum_{i \geqslant 0} b_i s^i && \in E_*(\underline{F}_*)[\![s]\!]. \end{aligned}$$

**Notation** For $n \in \mathbb{N}$, let $\pi_n = \frac{p^n - 1}{p-1}$.

**Proposition**

In $QE_*(\underline{F}_*)$, we have

$$v_n^E b_1^{\pi_n} = b_1^{\pi_{n+1}-1}[v_n^F].$$

**Sketch proof**

- Start from the Ravenel-Wilson relation in $E_*(\underline{F}_*)[\![s]\!]$:

$$b([p]_E(s)) = [p]_F(b(s)).$$

- Quotient to the additive world of $QE_*(\underline{F}_*)[\![s]\!]$.
- Reduce mod $p$.
- Equate powers of $s$ and read off identities.
- In the case $n = 1$, looking at leading terms leads directly to the required identity.
- Use this as the start of a recursive procedure to prove the general case.

The **algebraic suspension element** is $e = \eta_{1*}u_1 \in E_1(\underline{F}_1)$, where $u_1$ is the image in $E_1(S^1)$ of the unit in $E_0(S^0)$ under the suspension isomorphism $E_0(S^0) \cong E_1(S^1)$ and $\eta_1 : S^1 \to \underline{F}_1$ is the unit map.

The suspension map $E_{l-1}(\underline{F}_{k-1}) \to E_l(\underline{F}_k)$ induced by $\Sigma \underline{F}_{k-1} \to \underline{F}_k$ is, up to sign, $\circ$-multiplication by $e$.

Since we have $b_1 = -e^{\circ 2} \in E_2(\underline{F}_2)$, we can rewrite the proposition in terms of $e$ rather than $b_1$:

$$v_n^E e^{2\pi_n} = e^{2(\pi_{n+1}-1)}[v_n^F] = e^{2(p^n-1)+2\pi_n}[v_n^F].$$

**Definition**

The *$E$-additive loop height of $F$* is the least positive integer $h$ such that the identity

$$v_n^E e^h = e^{2(p^n-1)+h}[v_n^F].$$

holds in $QE_*(\underline{F}_*)$.

It follows from the proposition that

$$1 \leqslant h \leqslant 2\pi_n.$$

**Examples**

- From the Ravenel-Wilson calculation of $K(n)_*(\underline{BP}_*)$, the $K(n)$-additive loop height of $BP$ is $2\pi_n$.
- From Wilson's calculation of $K(n)_*(\underline{K(n)}_*)$, the self additive loop height of $K(n)$ is 1.

## Definition

The *$E$-unstable loop height of $F$* is the least positive integer $h$ such that the identity

$$v_n^E e^{\circ h} = e^{\circ 2(p^n-1)+h} \circ [v_n^F]$$

holds in $E_*(\underline{F}_*)$.

Since $\circ$-multiplication by $e$ factors via the additive co-operations, the $E$-unstable loop height of $F$ is at least the $E$-additive loop height and at most one more. In particular, $2\pi_n + 1$ is an upper bound.

## Example

The self unstable loop height of $K(n)$ is 1.

**Definition**

$s = (v_n^E)^{-1} e^{\circ 2(p^n - 1)} \circ [v_n^F] \quad \in E_0(\underline{F}_0)$.

**Proposition: splitting co-operations**

Let $h$ be the $E$-unstable loop height of $F$. Then

- $s \circ s = s$,
- $e^{\circ h} \circ s = e^{\circ h} = s \circ e^{\circ h}$,
- There is some $s'$ such that $e^{\circ h} \circ s' = s$.
- There is a split short exact sequence of graded algebras

$$0 \to sE_*(\underline{F}_*) \to E_*(\underline{F}_*) \to E_*(\underline{F}_*)/sE_*(\underline{F}_*) \to 0.$$

- $\Sigma^h$ is an isomorphism on $sE_*(\underline{F}_*)$ and zero on $(1-s)E_*(\underline{F}_*)$.

**Proposition cont.**

- Passing to colimits via suspension maps gives, for $k, l \in \mathbb{Z}$, $E_l(F) \cong sE_{k+l}(\underline{F}_k)$, so we have a "destabilization map" $E_l(F) \to E_{k+l}(\underline{F}_k)$, right inverse to stabilization. The image of this is the same as the image of the $h$-fold suspension $\Sigma^h : E_{k+l-h}(\underline{F}_k) \to E_{k+l}(\underline{F}_k)$.

Since we have assumed that we have good duality, dualizing all of the above gives the theorem.

In particular a map $\underline{K(n)}_k \to \underline{K(n)}_l$ is an infinite loop map if and only if it is a loop map.

## §2. Plethories

Tall and Wraith, *Representable functors and operations on rings*, Proc. London Math. Soc. 20, 1970.

Borger and Wieland, *Plethystic algebra*, Adv. Math. 194, 2005.

Boardman, Johnson and Wilson, *Unstable operations in generalised cohomology*, Handbook of algebraic topology, 1995.

**Slogan** For suitable cohomology theories $E$,

$E^*(\underline{E}_*)$ is a bigraded completed plethory and $E^*(X)$ is a graded completed algebra over this plethory.

This can be viewed as a monoidal reinterpretation of the following comonadic description of [BJW].

**Theorem** [BJW]

$E^*(\underline{E}_*)$ represents a comonad in FAlg and $E^*(X)$ is a comodule in FAlg for this comonad.

Here FAlg is the category of complete Hausdorff commutative filtered $E^*$-algebras.

This means:

- The hom functor FAlg $\to$ GSet

$$A^* \mapsto \mathrm{FAlg}(\mathrm{E}^*(\underline{\mathrm{E}}_*), \mathrm{A}^*)$$

  has a natural lift to a functor $U : \mathrm{FAlg} \to \mathrm{FAlg}$.

- There are natural transformations $\mu : U \to U^2$, $\varepsilon : U \to I$ satisfying coassociativity and counit conditions.

- There is a coaction map $\rho : E^*(X) \to U(E^*(X))$ satisfying the standard conditions.

**Notation** Let $k$ be a commutative associative unital ground ring and let $\mathcal{A}$ be the category of associative commutative unital $k$-algebras ("$k$-algebras" from now on). Write $\otimes$ for $\otimes_k$.

**Definition** A $(k-k)$-*biring* $B$ is a $k$-algebra object in $\mathcal{A}^{\mathrm{op}}$.

So, $B$ is a $k$-algebra together with $k$-algebra maps:

| | |
|---|---|
| $\Delta^+ : B \to B \otimes B$ | coaddition |
| $\varepsilon^+ : B \to k$ | cozero |
| $v : B \to B$ | additive inverse |
| $\Delta^\times : B \to B \otimes B$ | comultiplication |
| $\varepsilon^\times : B \to k$ | counit |

and a ring map $\beta : k \to \hom_{\mathcal{A}}(B, k)$, ($k$-colinear structure), satisfying axioms...

**Notation**

$$\begin{aligned} \Delta^+(b) &= \sum b^{(1)} \otimes b^{(2)}, \\ \Delta^\times(b) &= \sum b^{[1]} \otimes b^{[2]}. \end{aligned}$$

## Key Property

For a biring $B$, $\hom_{\mathcal{A}}(B, A)$ is again a $k$-algebra.

I.e. the functor $B_* = \hom_{\mathcal{A}}(B, -) : \mathcal{A} \to \mathrm{Sets}$ lifts to $\mathcal{A}$.

Here
$$(f+g)(b) = \sum f(b^{(1)})g(b^{(2)}),$$
$$(fg)(b) = \sum f(b^{[1]})g(b^{[2]}).$$

Indeed
$$\mathrm{Birings} \simeq \mathrm{CovRep}(\mathcal{A}, \mathcal{A}).$$

## Examples

- $k[e]$: this represents the identity functor $\mathcal{A} \to \mathcal{A}$. The element $e$ is "ring-like":
$$\begin{aligned} \Delta^{+}(e) &= 1 \otimes e + e \otimes 1, \\ \Delta^{\times}(e) &= e \otimes e. \end{aligned}$$
- More generally, the free algebra $S(X)$ on a set $X$ with each $x \in X$ ring-like.
- $\mathrm{Functions}(\mathbb{F}_p, \mathbb{F}_p)$.

## The monoidal structure on birings: the composition product

For a biring $B$ and a $k$-algebra $A$, there is a $k$-algebra $B \odot A$ such that

$$\hom_{\mathcal{A}}(A, \hom_{\mathcal{A}}(B, C)) \cong \hom_{\mathcal{A}}(B \odot A, C),$$

i.e.

$$(B \odot A)_* = A_* B_*.$$

If $A$ is itself a biring then so is $B \odot A$, with structure maps $1 \odot \Delta^+$, $1 \odot \Delta^\times$, etc, and the above bijection is then an isomorphism of algebras.

$\odot$ is a monoidal structure on the category $\mathcal{B}$ of $(k-k)$-birings with unit $k[e]$. It is associative, but not symmetric or bilinear (though it is linear in $B$), and it distributes over $\otimes$.

$B \odot A$ is the free $k$-algebra on symbols $b \odot a$ subject to relations:

$$
\begin{aligned}
(b_1 + b_2) \odot a &= b_1 \odot a + b_2 \odot a \\
b_1 b_2 \odot a &= (b_1 \odot a)(b_2 \odot a) \\
1 \odot a &= 1 \\
b \odot (a_1 + a_2) &= \sum (b^{(1)} \odot a_1)(b^{(2)} \odot a_2) \\
b \odot (a_1 a_2) &= \sum (b^{[1]} \odot a_1)(b^{[2]} \odot a_2) \\
b \odot (-a) &= v(b) \odot a \\
b \odot 1 &= \varepsilon^{\times}(b) \\
b \odot 0 &= \varepsilon^{+}(b)
\end{aligned}
$$

**Definition** [T-W, B-W]

A *$k$-plethory* $P$ is a monoid in $(\mathcal{B}, \odot)$, i.e. $P$ is a biring with maps of birings $\circ : P \odot P \to P$ and $k[e] \to P$ satisfying associativity and unitality.

A plethory is also known as a *Tall-Wraith monoid.*

Let $P$ be a $k$-plethory. A *$P$-algebra* $A$ is a $k$-algebra with an action of $P$, that is a map of $k$-algebras $P \odot A \to A$ such that the usual diagrams commute.

A $k$-plethory structure on a biring $P$ is the same as a monad structure on the functor $P \odot -$, and by adjunction, also the same as a comonad structure on the functor $\hom_{\mathcal{A}}(P, -)$. An action of $P$ on $A$ is the same as a comodule over the comonad.

From the construction of $\odot$, in a plethory $P$ we have formulas like

$$p \odot (qr) = \sum (p^{[1]} \odot q)(p^{[2]} \odot r).$$

That $\odot$ is a map of birings encodes other quite complicated formulas, for example for rewriting $\Delta^+(p \odot q)$ and $\Delta^\times(p \odot q)$.

**Examples**

- $k[e]$ and Functions$(\mathbb{F}_p, \mathbb{F}_p)$ are plethories, with biring structure as before, together with composition.
- Let $M$ be a monoid. The free algebra on the underlying set of $M$ is a plethory with $m \in M$ ring-like and composition from the monoid product $m_1 \odot m_2 \to m_1 m_2$.
- $\Lambda = \mathbb{Z}[\lambda_1, \lambda_2, \ldots]$, the ring of symmetric functions in infinitely many variables, where the $\lambda_i$ are the elementary symmetric functions. This is a plethory where $\odot$ is plethysm of symmetric functions. A ring $R$ is a $\Lambda$-algebra if and only if it is a $\lambda$-ring.

All this can be generalised to the category $\mathcal{V}$ of $V$-algebras, where $V$ is a variety of algebras (in the sense of universal algebra, specified by operations and identities).

For example, $\mathcal{V}$ = abelian groups or $k$-modules or $k$-algebras or ...

**Definition**

A co-$\mathcal{V}$-object in $\mathcal{V}$ is a $\mathcal{V}$-algebra object in $\mathcal{V}^{\mathrm{op}}$.

Write $\mathcal{V}\mathcal{V}^c$ for the category of co-$\mathcal{V}$-objects in $\mathcal{V}$.

**Theorem**

- The free functor $F_V : \mathrm{Set} \to \mathcal{V}$ lifts to $\mathcal{V}\mathcal{V}^c$.
- There is a pairing $\odot : \mathcal{V}\mathcal{V}^c \times \mathcal{V} \to \mathcal{V}$.
- $\mathcal{V}\mathcal{V}^c$ has a monoidal structure $\odot$, with unit $F_V(\{*\})$, the free $\mathcal{V}$-algebra on a one point set.
- For a monoid $M$ in Set, $F_V(|M|)$ is a monoid in $\mathcal{V}\mathcal{V}^c$.

## Examples

| $\mathcal{V}$ | $\mathcal{V}\mathcal{V}^c$ | $\odot$ | $\odot$-monoid $P$ | $P$-algebra |
|---|---|---|---|---|
| ab gp | ab gp | $\otimes_{\mathbb{Z}}$ | ring $R$ | $R$-module |
| $k$-mod | $k$-mod | $\otimes_k$ | $k$-algebra $A$ | $A$-module |
| $\mathcal{A}$ | birings | $\odot$ | plethory $P$ | $P$-algebra |

## Example

$E = K(1)$ for $p$ an odd prime

$E^0(\underline{E}_0)$ is a completed version of the free plethory on the submonoid of $\mathbb{N}_0$ under multiplication generated by 0 and $\tilde{q}$, where the generators correspond to $\Psi^0$ and $\Psi^{\tilde{q}}$.