Stable and unstable operations in mod p cohomology theories

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Based on joint work in progress with Andrew Stacey.

§1. Stable and unstable operations

We will discuss operations between two multiplicative graded cohomology theories $E^*(-)$ and $F^*(-)$ which are commutative and complex orientable.

There are various kinds of operations:

- **Stable operations** $F^*(-) \to E^*(-)$ of degree $l$, given by $E^l(F) = [F, E]^l$.
- **Unstable operations** $F^k(-) \to E^{k+l}(-)$, given by $E^{k+l}(F^k)$.
- **Additive operations** $F^k(-) \to E^{k+l}(-)$, given by $P E^{k+l}(F^k)$.

The restriction map from stable to unstable operations factors via additive operations, so for each $k \in \mathbb{Z}$ we have maps

$$E^l(F) \to P E^{k+l}(F^k) \hookrightarrow E^{k+l}(F^k).$$
We can also consider the corresponding objects in the world of co-operations.

- **Stable co-operations**, given by $E_l(F)$, $l \in \mathbb{Z}$.

- **Unstable co-operations**, given by $E_{k+l}(F_k)$, $k, l \in \mathbb{Z}$.

- **Additive co-operations**, given by $QE_{k+l}(F_k)$, the indecomposables of $E_{k+l}(F_k)$.

The stabilization map $\sigma_{k*} : E_{k+l}(F_k) \to E_l(F)$, induced by $\sigma_k : \Sigma^\infty F_k \to F$, factors via the additive co-operations:

$$E_{k+l}(F_k) \to QE_{k+l}(F_k) \to E_l(F).$$
Throughout $p$ will be an odd prime. We assume our theories satisfy the following conditions.

- $E^*$ has characteristic $p$.
- The formal group law of $E^*(-)$ has finite height, say $n$.
- The coefficient of the first term in the $p$-series for $E^*(-)$ is invertible.
- The various $E^*(-)$-modules of operations from $F^*(-)$ to $E^*(-)$ are the duals over $E^*$ to the corresponding $E^*$-modules of co-operations.

For example, we can take $E$ to be Morava $K$-theory $K(n)$ and $F$ to be any (multiplicative commutative complex orientable) theory.
Theorem

• (Bousfield, Kuhn) For each \(k, l \in \mathbb{Z}\), there’s a map \(\Delta^\infty : E^{k+l}(F_k) \to E^l(F)\), left-inverse to restriction.

• There is a positive integer \(h \leq 2\frac{p^n-1}{p-1} + 1\), such that an operation \(F^k(-) \to E^{k+l}(-)\) is a component of a stable operation if and only if it is the \(h\)-fold loop of an operation. Equivalently, a map \(F_k \to E_{k+l}\) is an infinite loop map if and only if it is an \(h\)-fold loop map.

• For \(\rho_k \in E^{k+l}(F_k)\), the components of \(\Delta^\infty(\rho_k)\) are given by:

\[
(\Delta^\infty(\rho_k))_m = (\nu_n^E)^{-i}(\Omega^j \rho_k)(\nu_n^F)^i,
\]

where \(i, j \geq 0\) are such that \(j \geq h\) and \(m - k = 2(p^n - 1)i - j\).
Let $x^E \in E^2(\mathbb{C}P^\infty)$ be the universal Chern class. Then $E^*(\mathbb{C}P^\infty) = E^*[x^E]$, 
$E_*(\mathbb{C}P^\infty) = E_*\{\beta_1^E, \beta_2^E, \ldots\}$ 
where $\beta_i^E$ is dual to $(x^E)^i$.

The $p$-series of $E$ is 
$[p]_E(x^E) = x^E +_E x^E +_E \cdots +_E x^E \in E^*[x^E]$. 
Modulo $p$ this has the form 

$$[p]_E(x^E) \equiv \sum_{i \geq 1}^E v_i^E(x^E)p^i,$$

where $v_i^E \in E^{-2(p^i-1)}$. The hypotheses on $E$ mean that 

$$[p]_E(x^E) \equiv v_n^E(x^E)p^n + \text{higher terms},$$

where $v_n^E \in E^{-2(p^n-1)}$ is invertible.

The universal Chern class $x^F$ represents a map $x^F : \mathbb{C}P^\infty \rightarrow F_2$. Consider the induced map $x_*^F : E_*(\mathbb{C}P^\infty) \rightarrow E_*(F_2)$ and write 

$$b_i = x_*^F(\beta_i^E) \in E_*(F_2),$$

$$b(s) = \sum_{i \geq 0} b_is^i \in E_*(F_2)[s].$$
**Notation**  For $n \in \mathbb{N}$, let $\pi_n = \frac{p^n - 1}{p - 1}$.

**Proposition**

In $Q E_\ast(F_\ast)$, we have

$$v_n^E b_1^{\pi_n} = b_1^{\pi_{n+1} - 1} [v_n^F].$$

**Sketch proof**

- Start from the Ravenel-Wilson relation in $E_\ast(F_\ast)[s]$:

  $$b([p]_E(s)) = [p]_F(b(s)).$$

- Quotient to the additive world of $Q E_\ast(F_\ast)[s]$.

- Reduce mod $p$.

- Equate powers of $s$ and read off identities.

- In the case $n = 1$, looking at leading terms leads directly to the required identity.

- Use this as the start of a recursive procedure to prove the general case.
The **algebraic suspension element** is
\[ e = \eta_1 \ast u_1 \in E_1(\underline{F}_1), \]
where \( u_1 \) is the image in \( E_1(S^1) \) of the unit in \( E_0(S^0) \) under the suspension isomorphism \( E_0(S^0) \cong E_1(S^1) \) and \( \eta_1 : S^1 \to \underline{F}_1 \) is the unit map.

The suspension map \( E_{l-1}(\underline{F}_{k-1}) \to E_l(\underline{F}_k) \) induced by \( \Sigma \underline{F}_{k-1} \to \underline{F}_k \) is, up to sign, \( \circ \)-multiplication by \( e \).

Since we have \( b_1 = -e \circ 2 \in E_2(\underline{F}_2) \), we can rewrite the proposition in terms of \( e \) rather than \( b_1 \):
\[
\nu_n^E e^{2\pi n} = e^{2(\pi n+1-1)}[\nu_n^F] = e^{2(p^n-1)+2\pi n}[\nu_n^F].
\]
**Definition**

The *E*-additive loop height of $F$ is the least positive integer $h$ such that the identity

$$v_n^E e^h = e^{2(p^n-1)+h} [v_n^F].$$

holds in $QE_* (F_* )$.

It follows from the proposition that

$$1 \leq h \leq 2\pi_n.$$

**Examples**

- From the Ravenel-Wilson calculation of $K(n)_*(BP_*)$, the $K(n)$-additive loop height of $BP$ is $2\pi_n$.

- From Wilson’s calculation of $K(n)_*(K(n)_*)$, the self additive loop height of $K(n)$ is 1.
**Definition**

The *E-unstable loop height of F* is the least positive integer \( h \) such that the identity

\[
v_n^E e^{\circ h} = e^{\circ 2(p^n-1)+h} \circ [v_n^F]
\]

holds in \( E_*(F_*) \).

Since \( \circ \)-multiplication by \( e \) factors via the additive co-operations, the *E-unstable loop height of F* is at least the *E-additive loop height* and at most one more. In particular, \( 2\pi_n + 1 \) is an upper bound.

**Example**

The self unstable loop height of \( K(n) \) is 1.
**Definition**

\[ s = (v_n^E)^{-1} e^{o2(p^n-1)} \circ [v_n^F] \in E_0(F_0). \]

**Proposition: splitting co-operations**

Let \( h \) be the \( E \)-unstable loop height of \( F \). Then

- \( s \circ s = s \),
- \( e^{oh} \circ s = e^{oh} = s \circ e^{oh} \),
- There is some \( s' \) such that \( e^{oh} \circ s' = s \).
- There is a split short exact sequence of graded algebras

\[ 0 \to sE_*(F_*) \to E_*(F_*) \to E_*(F_*)/sE_*(F_*) \to 0. \]

- \( \Sigma^h \) is an isomorphism on \( sE_*(F_*) \) and zero on \( (1-s)E_*(F_*) \).
Proposition cont.

- Passing to colimits via suspension maps gives, for \( k, l \in \mathbb{Z} \), \( E_l(F) \cong sE_{k+l}(\underline{F}_k) \), so we have a “destabilization map” \( E_l(F) \to E_{k+l}(\underline{F}_k) \), right inverse to stabilization. The image of this is the same as the image of the \( h \)-fold suspension \( \Sigma^h : E_{k+l-h}(\underline{F}_k) \to E_{k+l}(\underline{F}_k) \).

Since we have assumed that we have good duality, dualizing all of the above gives the theorem.

In particular a map \( K(n)_k \to K(n)_l \) is an infinite loop map if and only if it is a loop map.
§2. Plethories


**Slogan** For suitable cohomology theories $E$,

$E^*(E_*)$ is a bigraded completed plethory and $E^*(X)$ is a graded completed algebra over this plethory.

This can be viewed as a monoidal reinterpretation of the following comonadic description of [BJW].
Theorem [BJW]

$E^*(E_*)$ represents a comonad in FAlg and $E^*(X)$ is a comodule in FAlg for this comonad.

Here FAlg is the category of complete Hausdorff commutative filtered $E^*$-algebras.

This means:

- The hom functor $\text{FAlg} \to \text{GSet}$
  
  $$A^* \mapsto \text{FAlg}(E^*(E_*), A^*)$$

  has a natural lift to a functor $U : \text{FAlg} \to \text{FAlg}$.

- There are natural transformations $\mu : U \to U^2$, $\varepsilon : U \to I$ satisfying coassociativity and counit conditions.

- There is a coaction map $\rho : E^*(X) \to U(E^*(X))$ satisfying the standard conditions.
**Notation** Let $k$ be a commutative associative unital ground ring and let $\mathcal{A}$ be the category of associative commutative unital $k$-algebras ("$k$-algebras" from now on). Write $\otimes$ for $\otimes_k$.

**Definition** A $(k-\mathcal{A})$-biring $B$ is a $k$-algebra object in $\mathcal{A}^{\text{op}}$.

So, $B$ is a $k$-algebra together with $k$-algebra maps:

$\Delta^+: B \to B \otimes B$ \hspace{1em} \text{coaddition}

$\varepsilon^+: B \to k$ \hspace{1em} \text{cozero}

$\nu: B \to B$ \hspace{1em} \text{additive inverse}

$\Delta^\times: B \to B \otimes B$ \hspace{1em} \text{comultiplication}

$\varepsilon^\times: B \to k$ \hspace{1em} \text{counit}

and a ring map $\beta: k \to \text{hom}_\mathcal{A}(B, k)$, $(k$-colinear structure), satisfying axioms...

**Notation**

$$\Delta^+(b) = \sum b^{(1)} \otimes b^{(2)},$$

$$\Delta^\times(b) = \sum b^{[1]} \otimes b^{[2]}.$$
**Key Property**

For a biring $B$, $\text{hom}_A(B, A)$ is again a $k$-algebra.

I.e. the functor $B_* = \text{hom}_A(B, -) : A \to \text{Sets}$ lifts to $A$.

Here

$$(f + g)(b) = \sum f(b^{(1)})g(b^{(2)}),$$

$$(fg)(b) = \sum f(b^{[1]})g(b^{[2]}).$$

Indeed

$$\text{Birings} \simeq \text{CovRep}(A, A).$$

**Examples**

- $k[e]$: this represents the identity functor $A \to A$. The element $e$ is “ring-like”:

  $$\Delta^+(e) = 1 \otimes e + e \otimes 1,$$
  $$\Delta^x(e) = e \otimes e.$$  

- More generally, the free algebra $S(X)$ on a set $X$ with each $x \in X$ ring-like.

- Functions($\mathbb{F}_p, \mathbb{F}_p$).
The monoidal structure on birings: the composition product

For a biring $B$ and a $k$-algebra $A$, there is a $k$-algebra $B \circ A$ such that

$$\text{hom}_A(A, \text{hom}_A(B, C)) \cong \text{hom}_A(B \circ A, C),$$

i.e.

$$(B \circ A)_* = A_*B_*.$$

If $A$ is itself a biring then so is $B \circ A$, with structure maps $1 \circ \Delta^+, 1 \circ \Delta^\times$, etc, and the above bijection is then an isomorphism of algebras.

$\circ$ is a monoidal structure on the category $\mathcal{B}$ of $(k - k)$-birings with unit $k[e]$. It is associative, but not symmetric or bilinear (though it is linear in $B$), and it distributes over $\otimes$. 
\( B \circ A \) is the free \( k \)-algebra on symbols \( b \circ a \) subject to relations:

\[
\begin{align*}
(b_1 + b_2) \circ a &= b_1 \circ a + b_2 \circ a \\
b_1 b_2 \circ a &= (b_1 \circ a)(b_2 \circ a) \\
1 \circ a &= 1 \\
b \circ (a_1 + a_2) &= \sum (b^{(1)} \circ a_1)(b^{(2)} \circ a_2) \\
b \circ (a_1 a_2) &= \sum (b^{[1]} \circ a_1)(b^{[2]} \circ a_2) \\
b \circ (-a) &= v(b) \circ a \\
b \circ 1 &= \varepsilon^-(b) \\
b \circ 0 &= \varepsilon^+(b)
\end{align*}
\]
**Definition** [T-W, B-W]

A *k*-plethory $P$ is a monoid in $(\mathcal{B}, \odot)$, i.e. $P$ is a biring with maps of birings $\odot : P \odot P \to P$ and $k[e] \to P$ satisfying associativity and unitality.

A plethory is also known as a *Tall-Wraith monoid*.

Let $P$ be a $k$-plethory. A *$P$-algebra* $A$ is a $k$-algebra with an action of $P$, that is a map of $k$-algebras $P \odot A \to A$ such that the usual diagrams commute.

A $k$-plethory structure on a biring $P$ is the same as a monad structure on the functor $P \odot -$ , and by adjunction, also the same as a comonad structure on the functor $\text{hom}_A(P, -)$. An action of $P$ on $A$ is the same as a comodule over the comonad.
From the construction of $\odot$, in a plethory $P$ we have formulas like

$$p \odot (qr) = \sum (p^{[1]} \odot q)(p^{[2]} \odot r).$$

That $\odot$ is a map of birings encodes other quite complicated formulas, for example for rewriting $\Delta^+(p \odot q)$ and $\Delta^\times(p \odot q)$.

**Examples**

- $k[e]$ and $\text{Functions}(\mathbb{F}_p, \mathbb{F}_p)$ are plethories, with biring structure as before, together with composition.

- Let $M$ be a monoid. The free algebra on the underlying set of $M$ is a plethory with $m \in M$ ring-like and composition from the monoid product $m_1 \odot m_2 \to m_1 m_2$.

- $\Lambda = \mathbb{Z}[\lambda_1, \lambda_2, \ldots]$, the ring of symmetric functions in infinitely many variables, where the $\lambda_i$ are the elementary symmetric functions. This is a plethory where $\odot$ is plethysm of symmetric functions. A ring $R$ is a $\Lambda$-algebra if and only if it is a $\lambda$-ring.
All this can be generalised to the category $\mathcal{V}$ of $\mathcal{V}$-algebras, where $\mathcal{V}$ is a variety of algebras (in the sense of universal algebra, specified by operations and identities).

For example, $\mathcal{V} = $ abelian groups or $k$-modules or $k$-algebras or ...

**Definition**

A co-$\mathcal{V}$-object in $\mathcal{V}$ is a $\mathcal{V}$-algebra object in $\mathcal{V}^{\text{op}}$. Write $\mathcal{VV}^c$ for the category of co-$\mathcal{V}$-objects in $\mathcal{V}$.

**Theorem**

- The free functor $F_\mathcal{V} : \text{Set} \to \mathcal{V}$ lifts to $\mathcal{VV}^c$.
- There is a pairing $\odot : \mathcal{VV}^c \times \mathcal{V} \to \mathcal{V}$.
- $\mathcal{VV}^c$ has a monoidal structure $\odot$, with unit $F_\mathcal{V}(\{*\})$, the free $\mathcal{V}$-algebra on a one point set.
- For a monoid $M$ in Set, $F_\mathcal{V}(|M|)$ is a monoid in $\mathcal{VV}^c$. 
**Examples**

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**Example**

\[ E = K(1) \] for \( p \) an odd prime

\[ E^0(E_0) \] is a completed version of the free plethory on the submonoid of \( \mathbb{N}_0 \) under multiplication generated by 0 and \( \tilde{q} \), where the generators correspond to \( \Psi^0 \) and \( \Psi^{\tilde{q}} \).