Infinite sums of Adams operations

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§1. K-theory operations

Notation

K	complex K-theory spectrum
k	connective version
p	odd prime
$K_{(p)}$, $k_{(p)}$	p-local versions
G, g	Adams summand
q	primitive mod p^2
	(topological generator of $\mathbb{Z}_p^{\times}$)
$\hat{q} = q^{p-1}$	(topological generator of $1 + p\mathbb{Z}_p$)
Ψ^α	stable p-local Adams operation,
	$\alpha \in \mathbb{Z}_{(p)}$

We will discuss two different approaches to describing rings of operations in p-local K-theory.

- Adams operations
  - describe the elements explicitly as certain infinite sums involving Adams operations
  - leads to explicit formulas for structure maps, module structures, ...
- Actions on coefficients
  - the action on coefficients is faithful
  - rings of operations can be identified with complicated subrings of the infinite direct product $\prod_{n\geqslant 0}\mathbb{Z}_{(p)}$, described by systems of congruences

Definition

1. For $n \geqslant 0$, define elements $\hat{\varphi}_n \in g^0(g)$ by

$$\hat{\varphi}_n = \prod_{i=0}^{n-1} (\Psi^q - \hat{q}^i).$$

2. For $n \geqslant 0$, we define elements $\Phi_n \in K^0_{(p)}(K_{(p)})$ by

$$\Phi_n = \prod_{i=1}^{n} (\Psi^q - q^{(-1)^i \lfloor i/2 \rfloor}).$$

For example,

$$\hat{\varphi}_4 = (\Psi^q - 1)(\Psi^q - \hat{q})(\Psi^q - \hat{q}^2)(\Psi^q - \hat{q}^3),$$

$$\Phi_4 = (\Psi^q - 1)(\Psi^q - q)(\Psi^q - q^{-1})(\Psi^q - q^2).$$

Theorem 1 **[CCW, Lellmann]**

1. The elements of $g^0(g)$ can be expressed uniquely as infinite sums

$$\sum_{n\geqslant 0} a_n \hat{\varphi}_n,$$

where $a_n \in \mathbb{Z}_{(p)}$.

2. The elements of $K^0_{(p)}(K_{(p)})$ can be expressed uniquely as infinite sums

$$\sum_{n\geqslant 0} a_n \Phi_n,$$

where $a_n \in \mathbb{Z}_{(p)}$.

Definition

The *Gaussian polynomial* $\begin{bmatrix} n \\ j \end{bmatrix} \in \mathbb{Z}[t]$ is defined by

$$\begin{bmatrix} n \\ j \end{bmatrix} = \prod_{i=0}^{j-1} \frac{1 - t^{n-i}}{1 - t^{j-i}}.$$

- $\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + t^j \begin{bmatrix} n-1 \\ j \end{bmatrix}$.
- $\begin{bmatrix} n \\ j \end{bmatrix} = 0$ if $j > n$.
- Write $\begin{bmatrix} n \\ j \end{bmatrix}_a$ for the value of $\begin{bmatrix} n \\ j \end{bmatrix}$ at $t = a$.

Congruences for $g^0(g)$

Theorem 2 **[CCW]**

If $\varphi \in g^0(g)$ acts on $g_{2(p-1)i} = \mathbb{Z}_{(p)}$ as multiplication by $\mu_i \in \mathbb{Z}_{(p)}$, then

$$\sum_{i=0}^{n} (-1)^{n-i} \hat{q}^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{\hat{q}} \mu_i \equiv 0 \mod p^{\delta_p(n)},$$

for all $n \geqslant 0$, where $\delta_p(n) = n + \nu_p(n!)$.

Moreover every sequence satisfying these congruences arises from a unique stable operation.

Examples of the $g^0(g)$ congruences

$$\begin{aligned} \mu_1 - \mu_0 &\equiv 0 \quad \bmod p \\ \mu_2 - (1+\hat{q})\mu_1 + \hat{q}\mu_0 &\equiv 0 \quad \bmod p^2 \\ \mu_3 - (1+\hat{q}+\hat{q}^2)\mu_2 & \\ + \hat{q}(1+\hat{q}+\hat{q}^2)\mu_1 - \hat{q}^3\mu_0 &\equiv 0 \quad \bmod p^{3+\nu_p(3!)} \end{aligned}$$

For example : $\mathbf{p = 3, q = 2, \hat{q} = 2^2 = 4}$

$$\begin{aligned} \mu_1 - \mu_0 &\equiv 0 \quad \bmod 3 \\ \mu_2 - 5\mu_1 + 4\mu_0 &\equiv 0 \quad \bmod 3^2 \\ \mu_3 - 21\mu_2 + 84\mu_1 - 64\mu_0 &\equiv 0 \quad \bmod 3^4 \end{aligned}$$

§2. Bousfield's category of K-theory modules

In his work on the $K_{(p)}$-local category, Bousfield considers the category $\mathcal{A}(p)$ with objects modules M over the group-ring $R = \mathbb{Z}_{(p)}[\mathbb{Z}_{(p)}^\times]$ (where $j \in \mathbb{Z}_{(p)}^\times$ corresponds to Ψ^j) satisfying the conditions:

(i) If M is finitely generated over $\mathbb{Z}_{(p)}$, then

(a) each Ψ^j acts on $M \otimes \mathbb{Q}$ by a diagonalisable matrix whose eigenvalues are integer powers of j,

(b) for each $m \geqslant 1$, the action of $\mathbb{Z}_{(p)}^\times$ on $M/p^m M$ factors through the quotient homomorphism $\mathbb{Z}_{(p)}^\times \to (\mathbb{Z}/p^k)^\times$ for sufficiently large k.

(ii) For each $x \in M$, the submodule $Rx \subset M$ is finitely generated over $\mathbb{Z}_{(p)}$ and satisfies condition (i).

Notation

We write A for the ring $K^0_{(p)}(K_{(p)})$.

We filter A by the ideals

$$A_m = \left\{ \sum_{n \geqslant m} a_n \Phi_n : a_n \in \mathbb{Z}_{(p)} \right\}.$$

So $A = A_0 \supset A_1 \supset A_2 \supset \cdots$.

The ideal A_m consists of all operations which act trivially on the coefficient groups $\pi_{2i}(K_{(p)})$ for $-\frac{m}{2} < i < \frac{m+1}{2}$.

A is complete in the filtration topology.

Example

For any spectrum X, $K_0(X; \mathbb{Z}_{(p)})$ is a discrete continuous A-module.

That is, for each $x \in K_0(X; \mathbb{Z}_{(p)})$, there is some n such that $A_n x = 0$.

Theorem 3 **[CCW]**

Bousfield's category $\mathcal{A}(p)$ is isomorphic to the category of discrete continuous A-modules.

We hope our explicit information about the structure of A will allow simplification of Bousfield's work.

He shows that the objects of the $K_{(p)}$-local category correspond to pairs (M, κ), where $M \in \mathcal{A}(p)_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{A}(\mathrm{p})_*}(\mathrm{M}, \mathrm{M})$.

He doesn't classify the morphisms.

§3. Adams operations and cobordism

Theorem 4 [**GW**]

$$Z(BP^0(BP)) \cong g^0(g)$$

Ingredients in the proof

- We have Adams operations $\Psi^{\alpha}_{BP} \in BP^0(BP)$, for $\alpha \in \mathbb{Z}_{(p)}^{\times}$.
- Define $\hat{\varphi}_n^{BP} \in BP^0(BP)$, by replacing Ψ^q by Ψ^q_{BP}.

- We get an injective algebra map $\iota : g^0(g) \to BP^0(BP)$ by sending $\sum_{n=0}^{\infty} a_n \hat{\varphi}_n$ to $\sum_{n=0}^{\infty} a_n \hat{\varphi}_n^{BP}$.
- We want to show $\mathrm{Im}\,\iota = \mathrm{Z(BP^0(BP))}$.
- $Z(BP^0(BP)) = D_{BP}$, the diagonal operations, acting as multiplication by $\mu_i \in \mathbb{Z}_{(p)}$ on $BP_{2(p-1)i}$.
- We will explain that D_{BP} is characterised by a family of congruences. The final (hard) step is to compare the $g^0(g)$ congruences and the D_{BP} congruences and show that they have the same solution sets.

Congruences for BP

We can identify D_{BP} with a subring of $\prod_{n \in \mathbb{N}} \mathbb{Z}_{(p)}$, characterised by a family of congruences.

We have a commutative diagram

$$\begin{array}{ccc} BP_*(BP) & \xrightarrow{\bar{\theta}} & BP_* \\ \uparrow{\scriptstyle \eta_R} & \nearrow{\scriptstyle \theta_*} & \\ BP_* & & \end{array}$$

where $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ and $BP_*(BP) = BP_*[t_1, t_2, \ldots]$.

We may write $\bar{\theta}(t^\gamma) = \sum_\delta D^\gamma_\delta(\mu) v^\delta$, where $D^\gamma_\delta(\mu)$ is determined recursively as a finite rational linear combination of the μ_i. On the other hand, we have $D^\gamma_\delta(\mu) \in \mathbb{Z}_{(p)}$.

Examples of $BP^0(BP)$ congruences

p = 3

$$
\begin{aligned}
\mu_1 - \mu_0 &\equiv 0 \quad \textbf{mod } 3 \\
\mu_2 - 2\mu_1 + \mu_0 &\equiv 0 \quad \textbf{mod } 3^2 \\
\mu_3 - 3\mu_2 + 3\mu_1 - \mu_0 &\equiv 0 \quad \textbf{mod } 3^3 \\
\mu_4 - \mu_0 &\equiv 0 \quad \textbf{mod } 3 \\
3^2\bar{\pi}_1^3(\mu_4 - \mu_0) & \\
- \bar{\pi}_2(\mu_3 - 3\mu_2 + 3\mu_1 - \mu_0) &\equiv \mathbf{0} \quad \textbf{mod } 3^4
\end{aligned}
$$

$$\bar{\pi}_n = 1 - 3^{3^n - 1}$$

Theorem 5 [**GW**]

For all primes p,

$$Z(MU^0_{(p)}(MU_{(p)})) \cong k^0_{(p)}(k_{(p)})$$

and

$$Z(MU^0(MU)) \cong k^0(k).$$