Stable and unstable operations in p-local K-theory

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Based on joint work with Francis Clarke & Martin Crossley.

Notation

K	complex K-theory spectrum
k	connective version
p	odd prime
$K_{(p)}$, $k_{(p)}$	p-local versions
q	primitive mod p^2
	(topological generator of $\mathbb{Z}_p^\times$)
Ψ^α	unstable p-local Adams operation,
	$\alpha \in \mathbb{Z}_{(p)}$

Standard fact

Ψ^α is a stable p-local operation if and only if $\alpha \in \mathbb{Z}_{(p)}^\times$.

We will describe the rings of stable and unstable operations in p-local K-theory and discuss the relationship between them.

Notation

$A = K^0(K)$,	stable degree zero operations, periodic
$B = k^0(k)$,	stable degree zero operations, connective
C	unstable additive operations

We have

$$A \subset B \subset C.$$

We write $A_{(p)}$, $B_{(p)}$, $C_{(p)}$ for the corresponding rings of operations for p-local K-theory and we have

$$A_{(p)} \subset B_{(p)} \subset C_{(p)}.$$

Unstable operations

Definition Define elements $\mu_n \in C$, for $n \geqslant 0$, by

$$\mu_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \Psi^i.$$

Theorem [Adams]

$$C = \left\{ \sum_{n \geqslant 0} a_n \mu_n \;\middle|\; a_n \in \mathbb{Z} \right\}.$$

Idea of Proof:

Consider $C \to K^0(\mathbb{C}P^\infty) = \mathbb{Z}[[t]]$ given by $\theta \mapsto \theta(1+t)$, where $1+t$ is the Hopf bundle. Note that $\Psi^j \mapsto (1+t)^j$.

This is an additive isomorphism, with injectivity proved using the Splitting Principle and surjectivity by noting that $\mu_n \mapsto t^n$.

This exhibits the ring as a completion of the monoid ring $\mathbb{Z}[\mathbb{N}_0^{\mathrm{mult}}]$.

Stable operations

Definition For $n \geqslant 0$, define elements $\varphi_n \in B_{(p)}$ by

$$\varphi_n = \prod_{i=0}^{n-1} (\Psi^q - q^i).$$

Theorem [**CCW**]

$$B_{(p)} = \left\{ \sum_{n \geqslant 0} a_n \varphi_n \;\middle|\; a_n \in \mathbb{Z}_{(p)} \right\}.$$

Idea of Proof:

Exploit duality and a nice basis for the dual object. This dual is a ring of “stably numerical polynomials” and has a basis involving Gaussian polynomials.

Definition

The *Gaussian polynomial* $\begin{bmatrix} n \\ j \end{bmatrix} \in \mathbb{Z}[q]$ is defined by

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{\theta_j(q^n)}{\theta_j(q^j)},$$

where $\theta_n(X) \in \mathbb{Z}[X]$ is given by

$$\theta_n(X) = \prod_{i=0}^{n-1} (X - q^i).$$

Basic properties

- $\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}$.
- $\begin{bmatrix} n \\ j \end{bmatrix} = 0$ if $j > n$.

Relating stable and unstable operations

Proposition

$$\varphi_n = \sum_{j=n}^{q^n} \left(\sum_{i=0}^{n} (-1)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \binom{q^i}{j} \right) \mu_j .$$

The range of summation ensures that we get a sensible formula expressing an infinite sum of the φ_n as an infinite sum of the μ_n, thus making explicit the inclusion $B_{(p)} \to C_{(p)}$.

We can characterize the image by a (complicated!) family of congruences.

Theorem

Let $\mu = \sum_{n\geqslant 0} a_n \mu_n \in C_{(p)}$.

Then $\mu \in B_{(p)}$ if and only if

$$\sum_{n=0}^{m}\sum_{k=n}^{m}\sum_{i=0}^{n} a_n (-1)^{m+n-i-k} q^{\binom{m-k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \binom{n}{i} i^k \equiv 0 \bmod p^{\gamma_p(m)},$$

for all $m \geqslant 0$,

where

$\gamma_p(m) = \lfloor m/(p-1) \rfloor + \nu_p((\lfloor m/(p-1) \rfloor)!)$.

Filtrations

We have decreasing filtrations by ideals

$$B_{(p)}{}_m = \left\{ \sum_{n \geqslant m} a_n \varphi_n : a_n \in \mathbb{Z}_{(p)} \right\}$$

and

$$C_{(p)}{}_m = \left\{ \sum_{n \geqslant m} a_n \mu_n : a_n \in \mathbb{Z}_{(p)} \right\}$$

Both $B_{(p)}$ and $C_{(p)}$ are complete in the filtration topology.

It is often convenient to work mod Ψ^0 and we can do this without losing information, since

$$C = \Psi^0 C \oplus (1 - \Psi^0)C,$$

with $\Psi^0 C$ a free abelian group of rank 1 generated by Ψ^0.

Completions

Theorem

There is an injective ring homomorphism $B_{(p)}[\Psi^p] \to C_{(p)}/\langle \Psi^0 \rangle$ and $C_{(p)}/\langle \Psi^0 \rangle$ is isomorphic to the completion of $B_{(p)}[\Psi^p]$ with respect to the induced filtration.

The completion is necessary here: the polynomial extension of $B_{(p)}$ by Ψ^p is not enough.

For example, one can show that $\sum_{n \geqslant 0} \mu_{p^n} \notin B_{(p)}[\Psi^p]$.

Definition

Define $\rho_n \in \mathbb{Z}_{(p)}[\Psi^p, \Psi^q]$, for $n \geqslant 0$, by

$$\rho_n = (-1)^n \Psi^0 + \sum_{m=0}^{n} \sum_{k=1}^{n} \beta_{n,m}^k \Psi^{p^{\nu_p(k)}} \varphi_m,$$

where $\beta_{n,m}^k = (-1)^{n-k} \binom{n}{k} \frac{\theta_m(k/p^{\nu_p(k)})}{\theta_m(q^m)}$, for $n, m \geqslant 0$ and $k \geqslant 1$.

Theorem

The elements of $C_{(p)}$ are uniquely expressible as infinite sums $\sum_{n \geqslant 0} a_n \rho_n$, where $a_n \in \mathbb{Z}_{(p)}$.

There is an isomorphism of rings between $C_{(p)}/\langle \Psi^0 \rangle$ and a completion of $\mathbb{Z}_{(p)}[\Psi^p, \Psi^q]$.

Question

Is there a nice choice to replace ρ_n ?

p-adic operations

The situation becomes simpler with p-adic coefficients.

Theorem

There is an isomorphism of rings

$$C_p/\langle\Psi^0\rangle \cong B_p[[\Psi^p]].$$

Let G_p denote the Adams summand and let U_p denote the unstable additive operations for G_p.

Theorem

There is an isomorphism of rings

$$U_p/\langle\Psi^0\rangle \cong \mathbb{Z}_p[[\Psi^q - 1, \Psi^p]].$$