Stable and unstable operations in $p$-local $K$-theory

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Based on joint work with Francis Clarke & Martin Crossley.
Notation

$K$ complex $K$-theory spectrum
$k$ connective version
$p$ odd prime
$K_{(p)}, k_{(p)}$ $p$-local versions
$q$ primitive mod $p^2$
\hspace{1cm} \text{(topological generator of } \mathbb{Z}_p^\times \text{)}
\Psi^\alpha \text{ unstable } p\text{-local Adams operation, } \alpha \in \mathbb{Z}_{(p)}$

Standard fact

\Psi^\alpha \text{ is a } \textbf{stable } p\text{-local operation if and only if } 
\alpha \in \mathbb{Z}_{(p)}^\times.$
We will describe the rings of stable and unstable operations in $p$-local $K$-theory and discuss the relationship between them.

**Notation**

- $A = K^0(K)$, stable degree zero operations, periodic
- $B = k^0(k)$, stable degree zero operations, connective
- $C$ unstable additive operations

We have

$$A \subset B \subset C.$$

We write $A_{(p)}$, $B_{(p)}$, $C_{(p)}$ for the corresponding rings of operations for $p$-local $K$-theory and we have

$$A_{(p)} \subset B_{(p)} \subset C_{(p)}.$$
**Unstable operations**

**Definition** Define elements $\mu_n \in C$, for $n \geq 0$, by

$$\mu_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \Psi^i.$$ 

**Theorem** [Adams]

$$C' = \left\{ \sum_{n\geq 0} a_n \mu_n \mid a_n \in \mathbb{Z} \right\}.$$ 

**Idea of Proof:**

Consider $C \to K^0(\mathbb{C}P^\infty) = \mathbb{Z}[[t]]$ given by $\theta \mapsto \theta(1 + t)$, where $1 + t$ is the Hopf bundle. Note that $\Psi^j \mapsto (1 + t)^j$.

This is an additive isomorphism, with injectivity proved using the Splitting Principle and surjectivity by noting that $\mu_n \mapsto t^n$.

This exhibits the ring as a completion of the monoid ring $\mathbb{Z}[\mathbb{N}_0^{\text{mult}}]$. 

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Definition For $n \geq 0$, define elements $\varphi_n \in B(p)$ by

$$\varphi_n = \prod_{i=0}^{n-1} (\Psi^q - q^i).$$

Theorem [CCW]

$$B(p) = \left\{ \sum_{n \geq 0} a_n \varphi_n \mid a_n \in \mathbb{Z}_p \right\}.$$ 

Idea of Proof:

Exploit duality and a nice basis for the dual object. This dual is a ring of “stably numerical polynomials” and has a basis involving Gaussian polynomials.
Definition

The Gaussian polynomial $\begin{bmatrix} n \\ j \end{bmatrix} \in \mathbb{Z}[q]$ is defined by

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{\theta_j(q^n)}{\theta_j(q^j)},$$

where $\theta_n(X) \in \mathbb{Z}[X]$ is given by

$$\theta_n(X) = \prod_{i=0}^{n-1} (X - q^i).$$

Basic properties

- $\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^{j} \begin{bmatrix} n-1 \\ j \end{bmatrix}.$
- $\begin{bmatrix} n \\ j \end{bmatrix} = 0$ if $j > n.$
Relating stable and unstable operations

Proposition

\[ \varphi_n = \sum_{j=n}^{q^n} \left( \sum_{i=0}^{n} (-1)^{n-i} q^{(n-i)} \binom{n}{i} \binom{q^i}{j} \right) \mu_j. \]

The range of summation ensures that we get a sensible formula expressing an infinite sum of the \( \varphi_n \) as an infinite sum of the \( \mu_n \), thus making explicit the inclusion \( B_{(p)} \rightarrow C_{(p)} \).
We can characterize the image by a (complicated!) family of congruences.

**Theorem**

Let \( \mu = \sum_{n \geq 0} a_n \mu_n \in C_p \).

Then \( \mu \in B_p \) if and only if

\[
\sum_{n=0}^{m} \sum_{k=n}^{m} \sum_{i=0}^{n} a_n (-1)^{m+n-i-k} q^{(m-k)/2} \binom{m}{k} \binom{n}{i} i^k \equiv 0 \mod p^{\gamma_p(m)},
\]

for all \( m \geq 0 \),

where

\[
\gamma_p(m) = \left\lfloor m/(p-1) \right\rfloor + \nu_p((\left\lfloor m/(p-1) \right\rfloor)!).
\]
Filtrations

We have decreasing filtrations by ideals

\[ B_{(p)_m} = \left\{ \sum_{n \geq m} a_n \varphi_n : a_n \in \mathbb{Z}_{(p)} \right\} \]

and

\[ C_{(p)_m} = \left\{ \sum_{n \geq m} a_n \mu_n : a_n \in \mathbb{Z}_{(p)} \right\} \]

Both \( B_{(p)} \) and \( C_{(p)} \) are complete in the filtration topology.

It is often convenient to work mod \( \Psi^0 \) and we can do this without losing information, since

\[ C = \Psi^0 C \oplus (1 - \Psi^0) C, \]

with \( \Psi^0 C \) a free abelian group of rank 1 generated by \( \Psi^0 \).
Theorem

There is an injective ring homomorphism $B_{(p)}[\Psi^p] \to C_{(p)}/\langle \Psi^0 \rangle$ and $C_{(p)}/\langle \Psi^0 \rangle$ is isomorphic to the completion of $B_{(p)}[\Psi^p]$ with respect to the induced filtration.

The completion is necessary here: the polynomial extension of $B_{(p)}$ by $\Psi^p$ is not enough.

For example, one can show that

$$\sum_{n \geq 0} \mu p^n \notin B_{(p)}[\Psi^p].$$
Definition

Define $\rho_n \in \mathbb{Z}_{(p)}[\Psi^p, \Psi^q]$, for $n \geq 0$, by

$$
\rho_n = (-1)^n \Psi^0 + \sum_{m=0}^{n} \sum_{k=1}^{n} \beta_{n,m}^k \Psi^{p\nu_p(k)} \varphi_m,
$$

where $\beta_{n,m}^k = (-1)^{n-k} \binom{n}{k} \frac{\theta_m(k/p^{\nu_p(k)})}{\theta_m(q^m)}$, for $n, m \geq 0$ and $k \geq 1$.

Theorem

The elements of $C_{(p)}$ are uniquely expressible as infinite sums $\sum_{n \geq 0} a_n \rho_n$, where $a_n \in \mathbb{Z}_{(p)}$.

There is an isomorphism of rings between $C_{(p)}/\langle \Psi^0 \rangle$ and a completion of $\mathbb{Z}_{(p)}[\Psi^p, \Psi^q]$.

Question

Is there a nice choice to replace $\rho_n$?
The situation becomes simpler with $p$-adic coefficients.

**Theorem**

There is an isomorphism of rings

$$C_p/\langle \Psi^0 \rangle \cong B_p[[\Psi^p]].$$

Let $G_p$ denote the Adams summand and let $U_p$ denote the unstable additive operations for $G_p$.

**Theorem**

There is an isomorphism of rings

$$U_p/\langle \Psi^0 \rangle \cong \mathbb{Z}_p[[\Psi^q - 1, \Psi^p]].$$