The structure of $k^0_{(p)}(k_{(p)})$

Product

$$\varphi_r\varphi_s = \sum_{j=\max(r,s)}^{r+s} \frac{\theta_{r+s-j}(q^r)\theta_{r+s-j}(q^s)}{\theta_{r+s-j}(q^{r+s-j})}\,\varphi_j.$$

Coproduct

$$\Delta\varphi_n = \sum_{\substack{r,s\geqslant 0\\ r+s\leqslant n}} \frac{\theta_{r+s}(q^n)}{\theta_r(q^r)\theta_s(q^s)}\,\varphi_{n-r}\otimes\varphi_{n-s}.$$

More formulas in $k^0_{(p)}(k_{(p)})$

Expansion of φ_n as a polynomial in Ψ^q

$$\varphi_n = \sum_{j=0}^{n} (-1)^{n-j} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \Psi^{q^j}.$$

Expression for Ψ^j

For $j \in \mathbb{Z}_{(p)}^{\times}$,

$$\Psi^j = \sum_{n \geqslant 0} \frac{\theta_n(j)}{\theta_n(q^n)} \varphi_n.$$

In particular, for $i \in \mathbb{Z}$,

$$\Psi^{q^i} = \sum_{n \geqslant 0} \begin{bmatrix} i \\ n \end{bmatrix}_q \varphi_n.$$

Congruences for $g^0(g)$

Theorem

If $\varphi \in g^0(g)$ acts on $g_{2(p-1)i} = \mathbb{Z}_{(p)}$ as multiplication by $\mu_i \in \mathbb{Z}_{(p)}$, then

$$\sum_{i=0}^{n} (-1)^{n-i} \hat{q}^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_{\hat{q}} \mu_i \equiv 0 \mod p^{\delta_p(n)},$$

for all $n \geqslant 0$, where $\delta_p(n) = n + \nu_p(n!)$.

Moreover every sequence satisfying these congruences arises from a unique stable operation.

Examples of the $g^0(g)$ congruences

$$\begin{aligned}
\mu_1 - \mu_0 &\equiv 0 \mod p \\
\mu_2 - (1+\hat{q})\mu_1 + \hat{q}\mu_0 &\equiv 0 \mod p^2 \\
\mu_3 - (1+\hat{q}+\hat{q}^2)\mu_2 & \\
+ \hat{q}(1+\hat{q}+\hat{q}^2)\mu_1 - \hat{q}^3\mu_0 &\equiv 0 \mod p^{3+\nu_p(3!)}
\end{aligned}$$

For example : $\mathbf{p = 3, q = 2, \hat{q} = 2^2 = 4}$

$$\begin{aligned}
\mu_1 - \mu_0 &\equiv 0 \mod 3 \\
\mu_2 - 5\mu_1 + 4\mu_0 &\equiv 0 \mod 3^2 \\
\mu_3 - 21\mu_2 + 84\mu_1 - 64\mu_0 &\equiv 0 \mod 3^4
\end{aligned}$$

Examples of $BP^0(BP)$ congruences

p = 3

$$
\begin{aligned}
\mu_1 - \mu_0 &\equiv 0 \quad \mathbf{mod}\ 3 \\
\mu_2 - 2\mu_1 + \mu_0 &\equiv 0 \quad \mathbf{mod}\ 3^2 \\
\mu_3 - 3\mu_2 + 3\mu_1 - \mu_0 &\equiv 0 \quad \mathbf{mod}\ 3^3 \\
\mu_4 - \mu_0 &\equiv 0 \quad \mathbf{mod}\ 3 \\
3^2\bar{\pi}_1^3(\mu_4 - \mu_0) & \\
-\bar{\pi}_2(\mu_3 - 3\mu_2 + 3\mu_1 - \mu_0) &\equiv \mathbf{0} \quad \mathbf{mod}\ 3^4
\end{aligned}
$$

$$\bar{\pi}_n = 1 - 3^{3^n - 1}$$