

# Algebraic structures in topology

Prospects in Mathematics  
Manchester, December 2012

Sarah Whitehouse

University of Sheffield

# Plan

- 1 Overview
- 2 A very brief survey
- 3 The basic idea (topological)
- 4 The definition (algebraic)
- 5 An application: minimal models
- 6 Some recent work
- 7 [Topology in the UK, maths at Sheffield]

## Overview

I will be talking about  $A_\infty$  structures.

They are one example of an algebraic structure arising in topology, with applications to many areas.

They arise when one weakens the notion of **associativity** to some kind of **homotopy associativity**.

This example nicely illustrates the interactions between algebra and topology, as well as relations to other areas of mathematics.

# A very brief survey

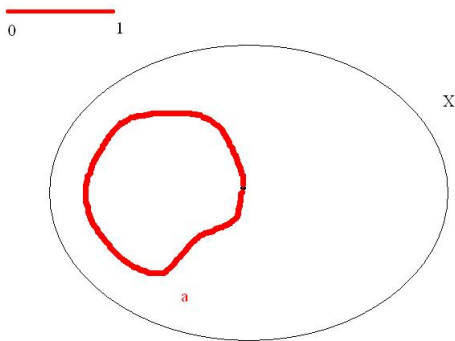
- Topology
  - 1960s [Stasheff]:  $A_\infty$ -spaces, key example is any loop space  $\Omega X$
  - stable homotopy theory, highly structured ring spectra
  - “brave new algebra”
  
- Algebra
  - $A_\infty$ -algebras, key examples are  $C_*(\Omega X)$ ,  $\text{Ext}_A^*(M, M)$
  - study of module categories, derived module categories  
1980s [Keller and others]
  - classification results for algebras

## A very brief survey, continued

- Mathematical physics
  - $A_\infty$ -categories, since 1990s [Fukaya, Kontsevich, ...]
  - key example is Fukaya category of a symplectic manifold
  - related to mirror symmetry
  
- Also higher category theory, geometry, ...

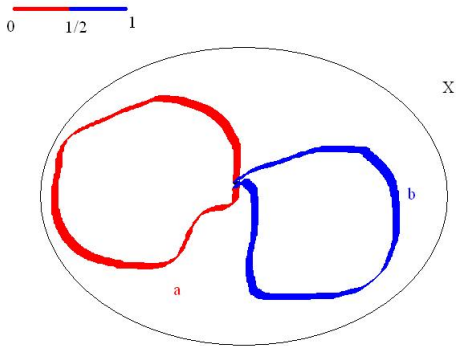
## The basic idea

Consider a 'multiplication' which is associative up to homotopy. For example, composition of based loops. A based loop in a based topological space  $(X, x_0)$  is a continuous map  $a : [0, 1] \rightarrow X$  such that  $a(0) = a(1) = x_0$ .



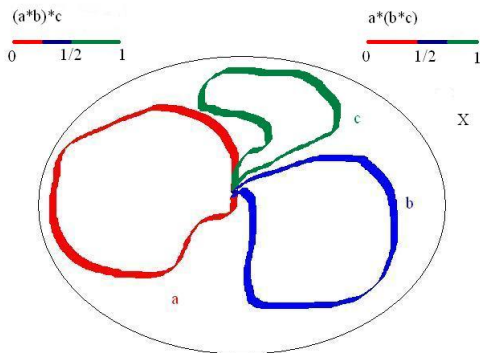
## Composition of loops

From two loops,  $a$  and  $b$ , we obtain a new loop  $a * b$  by 'going round  $a$  twice as fast and then  $b$  twice as fast'.



# Homotopy associativity of loop composition

As an immediate consequence of the way we compose loops, we find that composition is not strictly associative, but it is associative up to homotopy.





## Higher homotopy associativity

- Multiplication •

For each pair of points  $a, b$ , in  $Y = \Omega X$ , we have a single point  $a * b$ .

Multiplication is a map  $m_2 : Y \times Y \rightarrow Y$ .

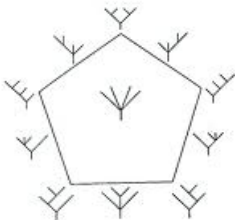


- Homotopy associativity ( $K_3$ )

For each triple of points  $a, b, c$ , we have the two points  $(a * b) * c$  and  $a * (b * c)$  and a path between them.

Thus the (naive) homotopy associativity of the multiplication gives a map  $m_3 : Y^3 \times K_3 \rightarrow Y$ .

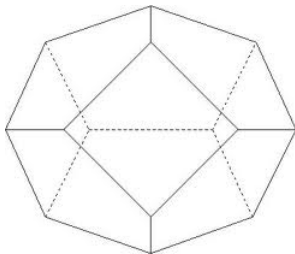
## A higher associativity condition ( $K_4$ )



Considering 4 points in  $Y$  and patching together the information from  $m_3$  we can define a map from the boundary of a pentagon to  $Y$ . So we get a map  $Y^4 \times \partial K_4 \rightarrow Y$ . Asking this map to extend over the interior of the pentagon is a higher homotopy associativity condition.

## The polytope $K_5$

There is an inductive procedure which continues this process: each time we can answer 'yes' to such a question, a new higher homotopy associativity condition presents itself. The next one involves the figure:



If the answer to all the questions is 'yes', we have an  $A_\infty$ -space. It has a multiplication which is homotopy associative in the strongest possible sense. This is the case for any loop space.

## The definition (algebraic)

### Definition

An  $A_\infty$ -algebra structure on a graded  $k$ -vector space  $A$  is a sequence of  $k$ -linear maps  $m_j : A^{\otimes j} \rightarrow A$  for  $j \geq 1$ , of degree  $j - 2$  such that, for each  $n \geq 1$ ,

$$\sum_{i,s} \pm m_{n+1-s}(1 \otimes \cdots \otimes m_s \otimes 1 \otimes \cdots \otimes 1) = 0.$$

## Low degrees and special cases

In particular,

- $m_1 : A \rightarrow A$  has degree  $-1$  and satisfies  $m_1 \circ m_1 = 0$ ; i.e. it is a differential on  $A$ .
- $m_1$  is a derivation with respect to  $m_2 : A^{\otimes 2} \rightarrow A$ .
- $m_3$  is chain homotopy associativity of  $m_2$ .
- The higher  $m$ s encode higher associativity conditions.
- If  $m_i = 0$  for all  $i \geq 3$ , then  $A$  is a differential graded associative algebra.
- If  $m_1 = 0$ , we say  $A$  is a **minimal**  $A_\infty$ -algebra. In this case, the multiplication is strictly associative, but we still have higher  $m$ s, encoding lots extra information.

## An application: minimal models

### Question:

What information about the homology  $H_*(A)$  of a differential graded algebra  $A$  do you need to recover  $A$ , up to quasi-isomorphism?

**Answer:** [Kadeishvili, Merkulov] (over a field)  
An  $A_\infty$ -algebra structure on  $H_*(A)$ .

A bit more precisely,  $H_*(A)$  admits a unique minimal  $A_\infty$ -structure in which  $m_2$  is induced by the multiplication on  $A$  and such that there is a quasi-isomorphism (of  $A_\infty$  algebras)  $H_*(A) \rightarrow A$ .

One can recover  $A$  from this structure.

## Some recent work

Recently Sagave defined *derived  $A_\infty$  algebras* in order to have a minimal model theorem which works over a general ground ring.

I have been involved in work to give a more conceptual approach to these structures.

This is an active area of current research.

# Topology in the UK

(continued on next page, apologies for any omissions)

- **Aberdeen**: Richard Hepworth, Ran Levi, Assaf Libman, Jarek Kedra
- **Cambridge**: Oscar Randal-Williams, Jacob Rasmussen, Burt Totaro
- **Durham**: Andrew Lobb, Vitaliy Kurlin, Dirk Schuetz,
- **Edinburgh**: Andrew Ranicki
- **Glasgow**: Andy Baker
- **Kent**: Constanze Roitzheim
- **Leicester**: John Hunton, Frank Neumann, Simona Paoli, Teimuraz Pirashvili



## Topology in the UK cont.

- **Manchester**: Nige Ray, Peter Symonds, Ted Voronov
- **Oxford**: Christopher Douglas, Marc Lackenby, Graeme Segal, Ulrike Tillmann
- **Sheffield**: John Greenlees, Neil Strickland, SW, Simon Willerton
- **Southampton**: Jelena Grbic, Ian Leary, Stephen Theriault
- **Swansea**: Martin Crossley, Jeff Giansiracusa
- **Warwick**: John Jones

# Pure Maths at Sheffield

**Algebra** commutative: tight closure, local cohomology;  
noncommutative: Weyl algebras, rings of differential  
operators, skew polynomial algebras

**Analysis** functional, harmonic, numerical and stochastic  
analysis

**Category theory** higher-dimensional category theory, model  
categories, triangulated categories, operads, applications

**Differential geometry** Lie groupoids and Lie algebroids,  
foliation theory, étale groupoids, orbifolds

**Number theory** elliptic curves, modular forms, Galois  
representations

**Topology** stable homotopy theory, equivariant versions,  
generalised cohomology rings, categorical foundations of  
homotopy theory

# Applied Maths at Sheffield

**Computational fluid dynamics** vortex dynamics, turbulence, engineering fluid dynamics, acoustic waves

**Environmental dynamics** synthetic aperture radar, turbulent diffusion, meteorology

**Mathematical biology** genetic epidemiology and statistical genetics, pattern formation

**Nonlinear control** adaptive backstepping control, the second-order sliding mode, nonlinear discrete-time systems

**Particle astrophysics and gravitation** cosmology, gravitation, classical and quantum behaviour of black holes, fundamental theory of space and time

**Solar physics and space plasma research centre** theoretical and observational issues, helioseismology, coronal-seismology, magnetohydrodynamics

# Probability and Statistics at Sheffield

**Bayesian Statistics** medical statistics, quantifying uncertainty in computer models, Bayesian time series analysis, applications to archaeology, ecology, health economics

**Statistical modelling and applied statistics** environmental statistics, scientific dating methods, calibration and model uncertainty, particle size distributions and pollution monitoring

**Probability** branching processes, random walks, fractals, random graphs, stochastic processes

# Applying to University of Sheffield

See <http://www.sheffield.ac.uk/maths/prospectivepg>

Sources of funding:

[EPSRC-funded Studentships](#) deadlines 1 February, 1 March, 5 April and 31 May 2013

[University Prize Scholarships](#) open to all, deadline 1 February 2013

[Graduate Teaching Assistantships](#) open to UK and EU applicants, deadlines 1 February, 1 March, 5 April and 31 May 2013

[SoMaS Studentships](#) open to UK and EU applicants, deadlines 1 February, 1 March, 5 April and 31 May 2013

Contact: Prof. Caitlin Buck, Director of Post-Graduate Research [c.e.buck@sheffield.ac.uk](mailto:c.e.buck@sheffield.ac.uk)