

**LMS-EPSRC SHORT COURSE ON ALGEBRAIC
 TOPOLOGY, SWANSEA, JULY 2005
 LECTURE COURSE 1: HOMOLOGY AND COHOMOLOGY
 THEORIES**

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PROBLEMS

Cohomology Theories

- (1) Formulate the definition of a natural transformation of homology theories. Let $h_*(-, -)$ and $k_*(-, -)$ be two homology theories satisfying the axioms (1) to (3) and the dimension axiom. Let $\varphi : h_*(-, -) \rightarrow k_*(-, -)$ be a natural transformation of homology theories. Prove that if $\varphi : h_0(pt) \rightarrow k_0(pt)$ is an isomorphism then $\varphi : h_n(X, A) \rightarrow k_n(X, A)$ is an isomorphism for each finite CW -pair (X, A) .

- (2) **Reduced theories.** Let CW^\bullet denote the category of based CW -complexes. A *reduced cohomology theory* \tilde{h}^* consists of a contravariant functor $h^* : CW^\bullet \rightarrow \mathcal{A}_*$ satisfying the following axioms.

- (a) (Homotopy) \tilde{h}^* is a homotopy functor;
 (b) (Exactness) For each based pair (X, A) there is a natural long exact sequence

$$\cdots \rightarrow \tilde{h}^n(X/A) \rightarrow \tilde{h}^n(X) \rightarrow \tilde{h}^n(A) \rightarrow \tilde{h}^{n+1}(X/A) \rightarrow \cdots;$$

- (c) (Wedge) $\tilde{h}^n(\bigvee_{\alpha \in I} X_\alpha) \cong \prod_{\alpha \in I} \tilde{h}^n(X_\alpha)$.

Check that $\tilde{h}^*(pt) = 0$. Check that the exact sequence in a reduced theory for the pair (CA, A) gives a suspension isomorphism $\sigma : \tilde{h}^n(A) \cong \tilde{h}^{n+1}(\Sigma A)$.

Given an unreduced theory h^* , show that one gets a reduced theory by setting $\tilde{h}^*(X) = h^*(X, x_0)$. Given a reduced theory \tilde{h}^* , show that one gets an unreduced theory by setting $h^*(X, A) = \tilde{h}^*(X/A)$. (In particular, $h^*(X) = \tilde{h}^*(X_+)$, where X_+ denotes the disjoint union of X with a point.)

- (3) **Mayer-Vietoris.** Let X be a CW -complex with subcomplexes A, B such that $X = A \cup B$. Show that we have a Mayer-Vietoris exact sequence

$$\cdots \rightarrow h_n(A \cap B) \rightarrow h_n(A) \oplus h_n(B) \rightarrow h_n(X) \rightarrow h_{n-1}(A \cap B) \rightarrow \cdots$$

for any homology theory h_* .

- (4) Check that bordism is an equivalence relation.

Extra Structure

- (1) Give an example of two spaces X and Y , such that $H^*(X)$ and $H^*(Y)$ are the same as graded abelian groups, but different as graded rings $H^*(X)$ and $H^*(Y)$.
- (2) Give an example of two spaces X and Y , such that $H^*(X; \mathbb{F}_2)$ and $H^*(Y; \mathbb{F}_2)$ are the same as graded rings, but different as modules over the Steenrod algebra.
- (3) Does there exist $n \in \mathbb{N}$ such that $\Sigma^n S^2 \vee S^4$ and $\Sigma^n \mathbb{C}P^2$ are homotopy equivalent?
- (4) Prove that if $\theta = \{\theta^n\}$ is a stable operation then each θ^n is additive.
- (5) From the definition, work out explicit formulas for the first few Adams operations in terms of exterior powers. Prove that $\Psi^k(x) = x^k$ if x is a line bundle and that $\Psi^k(x + y) = \Psi^k(x) + \Psi^k(y)$.
- (6) **Hopf invariant.** Let $f : S^{4n-1} \rightarrow S^{2n}$ and let $X = C_f$ be the mapping cone of f , that is, the space obtained from S^{2n} by attaching a $4n$ -cell using the map f .

Show that $\tilde{K}^0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, using the exact sequence of the pair (X, S^{2n}) .

Let x, y denote generators for the two copies of \mathbb{Z} and write $x^2 = hy$ for $h \in \mathbb{Z}$. (This h is the K -theory Hopf invariant of f .)

Suppose that h is odd. Show that $\Psi^2(x) \equiv y \pmod{2}$ and, using the relation $\Psi^2(\Psi^3(x)) = \Psi^3(\Psi^2(x))$, deduce that 2^n divides $3^n - 1$.

If $n = 2^r m$ with m odd, we write $\nu_2(n) = r$. Show that

$$\nu_2(3^n - 1) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ \nu_2(n) + 2, & \text{if } n \text{ is even.} \end{cases}$$

Deduce that if 2^n divides $3^n - 1$ then n must be 1, 2 or 4.

New theories from old

- (1) Let $E_*(-)$ be a homology theory and let X be a space or spectrum. Show that, if the Bousfield localization $L_E X$ exists, it is unique up to homotopy equivalence.
- (2) Check that the augmentation $MU_* \rightarrow \mathbb{Z} \subset \mathbb{Q}$ determined by $x_i \mapsto 0$ for all $i > 0$ makes \mathbb{Q} into an MU_* -module satisfying the conditions of LEFT, but that \mathbb{Z} does not satisfy the conditions.
- (3) Check that the MU_* -modules $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ and $E(p, n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$ satisfy the conditions of LEFT. Notice that $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$ does not.

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