LMS-EPSRC SHORT COURSE ON ALGEBRAIC TOPOLOGY, SWANSEA, JULY 2005
LECTURE COURSE 1: HOMOLOGY AND COHOMOLOGY THEORIES

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Problems

Cohomology Theories

(1) Formulate the definition of a natural transformation of homology theories. Let $h_*(-,-)$ and $k_*(-,-)$ be two homology theories satisfying the axioms (1) to (3) and the dimension axiom. Let $\varphi : h_*(-,-) \to k_*(-,-)$ be a natural transformation of homology theories. Prove that if $\varphi : h_0(pt) \to k_0(pt)$ is an isomorphism then $\varphi : h_n(X,A) \to k_n(X,A)$ is an isomorphism for each finite CW-pair (X,A).

(2) **Reduced theories.** Let $\mathcal{CW}^\bullet$ denote the category of based CW-complexes. A *reduced cohomology theory* $\tilde{h}^*$ consists of a contravariant functor $h^* : \mathcal{CW}^\bullet \to \mathcal{A}_*$ satisfying the following axioms.

(a) (Homotopy) $\tilde{h}^*$ is a homotopy functor;

(b) (Exactness) For each based pair (X,A) there is a natural long exact sequence
$$\cdots \to \tilde{h}^n(X/A) \to \tilde{h}^n(X) \to \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A) \to \dots;$$

(c) (Wedge) $\tilde{h}^n(\vee_{\alpha\in I} X_\alpha) \cong \prod_{\alpha\in I} \tilde{h}^n(X_\alpha)$.

Check that $\tilde{h}^*(pt) = 0$. Check that the exact sequence in a reduced theory for the pair (CA, A) gives a suspension isomorphism $\sigma : \tilde{h}^n(A) \cong \tilde{h}^{n+1}(\Sigma A)$.

Given an unreduced theory h^*, show that one gets a reduced theory by setting $\tilde{h}^*(X) = h^*(X, x_0)$. Given a reduced theory $\tilde{h}^*$, show that one gets an unreduced theory by setting $h^*(X,A) = \tilde{h}^*(X/A)$. (In particular, $h^*(X) = \tilde{h}^*(X_+)$, where X_+ denotes the disjoint union of X with a point.)

(3) **Mayer-Vietoris.** Let X be a CW-complex with subcomplexes A, B such that $X = A \cup B$. Show that we have a Mayer-Vietoris exact sequence
$$\cdots \to h_n(A\cap B) \to h_n(A) \oplus h_n(B) \to h_n(X) \to h_{n-1}(A\cap B) \to \dots$$
for any homology theory h_*.

(4) Check that bordism is an equivalence relation.

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Extra Structure

(1) Give an example of two spaces X and Y, such that $H^*(X)$ and $H^*(Y)$ are the same as graded abelian groups, but different as graded rings $H^*(X)$ and $H^*(Y)$.
(2) Give an example of two spaces X and Y, such that $H^*(X;\mathbb{F}_2)$ and $H^*(Y;\mathbb{F}_2)$ are the same as graded rings, but different as modules over the Steenrod algebra.
(3) Does there exist $n \in \mathbb{N}$ such that $\Sigma^n S^2 \vee S^4$ and $\Sigma^n \mathbb{C}P^2$ are homotopy equivalent?
(4) Prove that if $\theta = \{\theta^n\}$ is a stable operation then each θ^n is additive.
(5) From the definition, work out explicit formulas for the first few Adams operations in terms of exterior powers. Prove that $\Psi^k(x) = x^k$ if x is a line bundle and that $\Psi^k(x+y) = \Psi^k(x) + \Psi^k(y)$.
(6) **Hopf invariant.** Let $f : S^{4n-1} \to S^{2n}$ and let $X = C_f$ be the mapping cone of f, that is, the space obtained from S^{2n} by attaching a $4n$-cell using the map f.

Show that $\tilde{K}^0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, using the exact sequence of the pair (X, S^{2n}).

Let x, y denote generators for the two copies of $\mathbb{Z}$ and write $x^2 = hy$ for $h \in \mathbb{Z}$. (This h is the K-theory Hopf invariant of f.)

Suppose that h is odd. Show that $\Psi^2(x) \equiv y \pmod 2$ and, using the relation $\Psi^2(\Psi^3(x)) = \Psi^3(\Psi^2(x))$, deduce that 2^n divides $3^n - 1$.

If $n = 2^r m$ with m odd, we write $\nu_2(n) = r$. Show that

$$\nu_2(3^n - 1) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ \nu_2(n) + 2, & \text{if } n \text{ is even.} \end{cases}$$

Deduce that if 2^n divides $3^n - 1$ then n must be $1, 2$ or 4.

New theories from old

(1) Let $E_*(-)$ be a homology theory and let X be a space or spectrum. Show that, if the Bousfield localization $L_E X$ exists, it is unique up to homotopy equivalence.
(2) Check that the augmentation $MU_* \to \mathbb{Z} \subset \mathbb{Q}$ determined by $x_i \mapsto 0$ for all $i > 0$ makes $\mathbb{Q}$ into an MU_*-module satisfying the conditions of LEFT, but that $\mathbb{Z}$ does not satisfy the conditions.
(3) Check that the MU_*-modules $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ and $E(p,n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$ satisfy the conditions of LEFT. Notice that $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$ does not.

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