LMS-EPSRC SHORT COURSE ON ALGEBRAIC TOPOLOGY
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Lectures on Cohomology Theories: Outline

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Summary

The main tools of algebraic topology are cohomology theories. This lecture series will begin by defining these theories and surveying some of the most important examples, including ordinary cohomology, K-theory and cobordism. Since cohomology theories are given by spectra, there will be considerable interaction between this lecture series and the parallel series on spectra.

All the theories of interest come with useful extra structure and so products, operations and cooperations for cohomology theories will be covered. Some techniques for constructing new cohomology theories from existing ones will be discussed, including Bousfield localization and the Landweber exact functor theorem. With many cohomology theories to hand, one would like to be able to understand the relationships between them, and so we will give a brief introduction to the chromatic viewpoint. This uses the hierarchy of Morava K-theories to provide a way to organize the wealth of information provided by many different theories. If time permits, the lecture series will close with a discussion of which theories are even more highly structured, ending with an open problem.

It is, of course, impossible to be comprehensive. Our aims are to give a broad overview of the landscape and to equip the students with some examples that illustrate the general pattern of the theory.

1. Cohomology theories

1.1. **General philosophy of algebraic topology.** Extracting topological information using algebraic invariants. Examples.

1.2. **Formalizing the philosophy.** Functors from various topological categories to various algebraic ones. Homotopy functors. Examples: ordinary cohomology, homotopy.

1.3. **Definition of a cohomology theory.** The axioms, for homology and cohomology. Brief discussion of why these axioms are sensible and of why it's good to have both homology and cohomology.

1.4. **Examples.** The main examples will be ordinary cohomology, K-theory and cobordism. Mention will also be made of Morava K-theories, elliptic theories and stable homotopy.

2. Theories with extra structure : products and operations

2.1. **Why products? Why operations?** Brief discussion of why extra structure is useful.

2.2. **Introducing Coefficients.** Motivate working one prime at a time. Reminder about ordinary cohomology with coefficients in a ring R. Mention that there is a procedure for doing this for any theory.

2.3. **Products.** Reminder about cup product in ordinary cohomology. K-theory as example of geometric origin of products: tensor products of vector bundles. Mention of other examples.

2.4. **Operations.** Definition of unstable and stable cohomology operations. Examples: mention of Steenrod operations, emphasis on Adams operations in K-theory. E-cohomology of any space or spectrum is a module over the ring of operations.

2.5. **Cooperations.** Philosophy of the switch to cooperations. E-homology of any space or spectrum is a comodule over $E_*(E)$. Examples: dual Steenrod algebra, Landweber-Novikov algebra. Mention Adams SS.

3. New cohomology theories from old

3.1. **Splittings.** Introduction of coefficients may simplify things. Examples: Adams summand and BP.

3.2. **Localization.** Definition of E_*-local objects. Existence and properties of Bousfield localizations. Examples.

3.3. **Formal group laws and Landweber exact theories.** Definition of a formal group law. Cobordism and the universal formal group law. Building new theories out of cobordism from algebraic data: the Landweber exact functor theorem. Examples including Conner-Floyd, recovering K-theory.

4. Relationships between cohomology theories

4.1. **Extracting information from cohomology theories.** Main tool: spectral sequences, especially AHSS and ASS; refer to John McCleary's lectures.

4.2. **How much information is in a theory?** Trade-off between computability and information content. Examples: the Kunneth Theorem for Morava K-theories makes these computable; the coefficients for stable homotopy are not known.

4.3. **Some maps between theories.** Chern character. Maps from splittings. Maps from LEFT.

4.4. **The chromatic viewpoint.** How can we understand the relationships between our many examples of cohomology theories? The hierarchy of Morava K-theories. The chromatic convergence theorem.

5. More structure and an open problem

If time permits...

A very brief introduction to highly structured theories, beginning simply with homotopy associativity at the level of spaces. The old and new terminology for highly structured spectra. Mention of the various obstruction theories for detecting extra structure. Examples.