LMS-EPSRC SHORT COURSE ON ALGEBRAIC TOPOLOGY, SWANSEA, JULY 2005
LECTURE COURSE 1: HOMOLOGY AND COHOMOLOGY THEORIES

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Plan

1. Cohomology theories:	definition, examples including ordinary cohomology, K-theory and cobordism.
2. Products and operations:	motivation, definitions and examples (especially K-theory).
3. New theories from old:	splittings, Bousfield localization, formal group laws and the Landweber exact functor theorem.
4. Relationships between theories:	some natural transformations, the chromatic viewpoint.
5. More structure:	motivation, introduction to highly structured theories.

Preamble

One role of the Examples sessions is to allow you to ask questions. Details not given in the lectures can be filled-in there and if you have not seen some of the background material that is assumed, you can ask about it in these sessions.

References

Cohomology theories	Switzer,	Algebraic Topology - Homology and Homotopy
	Adams,	Stable homotopy and generalised homology
Ordinary cohomology	Hatcher,	Algebraic Topology
	Vick,	Homology theory
K-theory	Ayitah,	K-theory
	Hatcher,	Vector bundles and K-theory (pre-book)
Cobordism	Switzer again	
Localization, formal group laws, chromatic viewpoint	Ravenel,	Nilpotence and periodicity in stable homotopy theory

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1. Cohomology Theories

1.1. General philosophy of algebraic topology. Algebraic topology aims to study topological objects using algebraic invariants.

$$\text{Topology} \quad \rightsquigarrow \quad \text{Algebra}$$

Examples that you have probably already encountered include the genus of a surface and the Euler characteristic of a finite complex:

$$\begin{array}{ccc} \text{compact connected surfaces} & \rightsquigarrow & \mathbb{N} \\ S & \mapsto & \text{genus of } S \end{array}$$

and

$$\begin{array}{ccc} \text{finite complexes} & \rightsquigarrow & \mathbb{Z} \\ X & \mapsto & \chi(X) \end{array}$$

The essential idea of the whole subject is to use such invariants to distinguish different things: getting two different answers in algebra on the right-hand side means you must have started with two different things in topology on the left-hand side.

There are a couple of ways in which we will want to enrich the picture given by these examples. Firstly, we certainly want to distinguish not just objects but also maps between objects. Secondly, numerical invariants like the above only get you so far; we will want more algebraic structure on the right-hand side, such as a group or ring for each object rather than a number.

1.2. Formalizing the philosophy. The extraction of information is done by means of a *functor* from a topological *category* to an algebraic one. We assume that you have met at least the intuitive idea of categories, functors and natural transformations before.

We will use the following notation:

- $\mathcal{CW}$ the category of CW-complexes and maps,
- $\mathcal{CW}^2$ the category of CW-pairs and maps of pairs,
- $\mathcal{A}_*$ the category of $\mathbb{Z}$-graded abelian groups and homomorphisms of $\mathbb{Z}$-graded abelian groups.

(If you have not met CW-complexes, then you can just think of "nice spaces", such as manifolds.)

We will usually be interested in *homotopy functors*, mainly because *computable* functors tend to have this property. Of course, using homotopy functors we can only hope to distinguish spaces up to homotopy equivalence. But this is difficult enough; except in very special cases, it is unreasonable to hope for more anyway.

Examples.

Ordinary cohomology. We assume that you have met some version of "ordinary" cohomology before. This might be simplicial, singular, cellular or de Rham cohomology. It is a contravariant homotopy functor

$$\begin{array}{ccc} \mathcal{CW}^2 & \to & \mathcal{A}_* \\ (X,A) & \mapsto & H^*(X,A) \\ f:(X,A)\to(Y,B) & \mapsto & f^*:H^*(Y,B)\to H^*(X,A). \end{array}$$

Homotopy. For each $n \geqslant 0$, we have a covariant homotopy functor $\pi_n(-)$, taking values in abelian groups for $n \geqslant 2$, in groups for $n = 1$ and in sets for $n = 0$,

$$\begin{array}{ccc} X & \mapsto & \pi_n(X) = [S^n, X] \\ f : X \to Y & \mapsto & f_* : [S^n, X] \to [S^n, Y] \\ & & [g] \mapsto [f \circ g]. \end{array}$$

1.3. **Definition of a cohomology theory.** In practice the functors which are most useful share many of the properties of ordinary cohomology. So we come to the fundamental definition for this course, the axioms for a cohomology theory.

Definition 1.3.1. An *unreduced cohomology theory* $h^*(-,-)$ consists of a contravariant functor $h^* : \mathcal{CW}^2 \to \mathcal{A}_*$ satisfying the following axioms.

(1) (Homotopy) h^* is a homotopy functor, that is, if $f \simeq g$ then $h^*(f) = h^*(g)$;
(2) (Exactness) For each pair (X, A) and each $n \geqslant 0$ there is a map $\delta = \delta_n(X, A) : h^n(A) \to h^{n+1}(X, A)$ and a natural long exact sequence

$$\cdots \to h^n(X, A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} h^{n+1}(X, A) \to \dots ,$$

where i^*, j^* are induced by the inclusions $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$. Naturality means that given a map $f : (X, A) \to (Y, B)$, the following diagram commutes.

$$\begin{array}{ccccccccccc} \dots & h^n(X,A) & \xrightarrow{j^*} & h^n(X) & \xrightarrow{i^*} & h^n(A) & \xrightarrow{\delta} & h^{n+1}(X,A) & \to & \dots \\ & f^* \uparrow & & f^* \uparrow & & f^* \uparrow & & f^* \uparrow & & \\ \dots & h^n(Y,B) & \xrightarrow{j^*} & h^n(Y) & \xrightarrow{i^*} & h^n(B) & \xrightarrow{\delta} & h^{n+1}(Y,B) & \to & \dots \end{array}$$

(3) (Excision) If $U \subset A$ with $\overline{U} \subset A^o$, the inclusion of pairs $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $h^n(X, A) \to h^n(X \setminus U, A \setminus U)$ for all $n \in \mathbb{Z}$.
(4) $h^n(\amalg_{\alpha \in I} X_\alpha) \cong \prod_{\alpha \in I} h^n(X_\alpha)$.

Here, $h^n(X)$ means $h^n(X, \emptyset)$.

There is a corresponding definition of an *unreduced homology theory*, $h_*(-,-)$. This closely parallels the cohomology version, but in (1) we have a *covariant* functor, with maps consequently going in the opposite direction in (2) and (3), and with (4) replaced by $h_n(\amalg_{\alpha \in I} X_\alpha) = \oplus_{\alpha \in I} X_\alpha$.

Notes.
(1) The terms *extraordinary cohomology theory* and *generalized cohomology theory* are also used where we just use cohomology theory.
(2) Ordinary cohomology satisfies the axioms. Together with the *dimension axiom* giving the cohomology of a point,

$$H^n(pt) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

ordinary cohomology is characterized by properties (1) to (3) on finite CW-complexes. These are the Eilenberg-Steenrod axioms. Axiom (4) was introduced by Milnor. Ordinary cohomology is characterized by axioms (1) to

(4) together with the dimension axiom on all CW-complexes. (To allow for more general spaces a further weak equivalence axiom may be introduced.)

It happens that ordinary cohomology is concentrated in non-negative degrees, but we will soon see examples where this is not the case.

(3) Various equivalent formulations of the axioms are possible. In particular, there is a reduced version, which is more appropriate for based spaces. One gets a reduced cohomology theory $\tilde{h}^*(-)$ from an unreduced one by setting $\tilde{h}^*(X) = h^*(X, x_0)$. For a reduced theory, we have $\tilde{h}^*(X, A) \cong \tilde{h}^*(X/A)$, so we can avoid pairs of spaces. (See the problems.)

(4) For a reduced theory $\tilde{h}^*(X)$, consideration of the exact sequence associated to the pair (CA, A) gives us a *suspension isomorphism*, $\sigma : \tilde{h}^n(A) \cong \tilde{h}^{n+1}(\Sigma A)$. One can formulate the axioms in terms of σ rather than δ.

(5) Such theories come from topological objects called *spectra*, as will be explained in John Hunton's course. One consequence is that you always get a pair of related "dual" theories, homology and cohomology.

(6) The values of a cohomology theory on a point are known as the *coefficient groups* of the theory. Often we write h^* and h_* instead of $h^*(pt)$ and $h_*(pt)$. Up to a change of grading, in any given theory the cohomology and homology of a point are the same, $h_* = h^{-*}$. If one works with reduced theories, then the coefficients are given by the value on the zero sphere $S^0 = pt_+$ rather than on a point.

(7) It is good to have both homology and cohomology. They are closely related, but in general it is not possible to get one directly from the other in a simple way. They have different structure. For example, the ordinary cohomology of a space is a ring. As we will see, some examples, including K-theory, arise naturally as cohomology theories. Others, including bordism and stable homotopy, arise naturally as homology theories.

(8) These axioms give rise to invariants which are both interesting and *calculable* (especially by the methods of *spectral sequences* to be described in John McCleary's course).

(9) There are lots of interesting examples.

1.4. **Examples.** We will list some of the most important examples. It would be possible to devote an entire lecture course to any one of them. Here we hope to say enough to serve as a reminder to those who have encountered a theory before and to give a flavour to those who haven't.

(1) **Ordinary (co)homology.** As we have said, we assume some version of ordinary (co)homology has already been met. The various theories all agree on suitable categories of spaces. All involve associating a chain complex $C_*(X)$ to a space X. For example, for singular homology, $C_n(X)$ is the free abelian group with one generator for each singular n-simplex $\sigma : \Delta^n \to X$; the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \,|\, [v_0, \ldots, \widehat{v_i}, \ldots, v_n].$$

One then takes homology or cohomology of this chain complex, that is

$$H_n(X) = H_n(C_*(X)) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}$$

and $H^n(X) = H^n(\mathrm{Hom}(C_*(X), \mathbb{Z}))$. As noted above, the cohomology of a point is given by

$$H^n(pt) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

(2) **K-theory.** I will write K for complex K-theory; KU is also used. This cohomology theory is built out of *vector bundles* (at least for nice spaces). This gives the theory a geometric flavour, with natural examples of vector bundles given by tangent bundles to manifolds.

Recall that $p : E \to X$ is a *complex vector bundle* over X if each fibre $E_x = p^{-1}(x)$ has the structure of a finite dimensional complex vector space and a local triviality condition is satisfied. There are natural notions of homomorphism and isomorphism of vector bundles and we define Vect(X) to be the set of isomorphism classes of complex vector bundles over X. This is an abelian semigroup with operation + coming from direct sum of vector bundles. Recall also that there is a universal construction that associates an abelian group to any abelian semi-group, by formally adjoining inverse elements. So, for example, $\mathbb{Z}$ is associated to $\mathbb{N}$.

Definition 1.4.1. $K^0(X)$ is the universal group associated to the abelian semi-group Vect(X).

This means that the elements of $K^0(X)$ are formal differences of isomorphism classes of vector bundles.

Given a map $f : X \to Y$ and a vector bundle ξ over Y, we have a *pullback bundle* $f^*(\xi)$ over X. Passing to isomorphism classes and formal differences in the obvious way, we get $f^* : K^0(Y) \to K^0(X)$, making $K^0(-)$ into a contravariant functor from (say compact Hausdorff) spaces to abelian groups.

Next one proves that $K^0(-)$ is a homotopy functor. Then one wants to extend to a cohomology theory $K^*(-)$. One can use suspension to define $\tilde{K}^{-n}(X)$ as $\tilde{K}^0(\Sigma^n X)$, for $n \in \mathbb{N}$. It turns out that the resulting functors have a remarkable periodicity property : $K^0(X) \cong K^{-2}(X)$. Exploiting this *Bott periodicity*, we arrive at a 2-periodic cohomology theory, $K^*(-)$ with $K^n(X) \cong K^{n-2}(X)$ for all $n \in \mathbb{Z}$.

On a point, we find that

$$K^n(pt) = \begin{cases} \mathbb{Z}, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

In fact, we are actually getting a graded ring here and it is more illuminating to write

$$K^*(pt) = \mathbb{Z}[t, t^{-1}],$$

where $|t| = -2$.

If the above constructions are done with *real* vector bundles we get real K-theory, $KO^*(X)$. There are other variants.

(3) **Cobordism.** This time we will describe a homology theory, $MU_*(-)$, built from considering how suitable manifolds map into a space X.

Definition 1.4.2. Let M_1 and M_2 be smooth closed n-dimensional manifolds. Let $f_i : M_i \to X$ be continuous maps, for $i = 1, 2$. These maps are *bordant* if there is a map $f : W \to X$, where W is a smooth manifold of dimension $n+1$ whose boundary is the disjoint union of M_1 and M_2, such that the restriction of f to M_i is f_i. The map f is called a *bordism* between f_1 and f_2.

Bordism is an equivalence relation.

Definition 1.4.3. The set of bordism classes, $MO_*(X)$, is a group under disjoint union, the *n-th (unoriented) bordism group* of X.

We can make $MO_*(-)$ into a covariant functor $\mathcal{CW} \to \mathcal{A}_*$: given $f : X \to Y$ we define $f_* : MU_*(X) \to MU_*(Y)$ by $[M, f] \to [M, f \circ g]$. Homotopy invariance is not hard - a homotopy gives a bordism: if $f_0 \simeq_F f_1 : X \to Y$ then a bordism between $(f_0)_*[M, g] = [M, f_0 \circ g]$ and $(f_1)_*[M, g] = [M, f_1 \circ g]$ is given by $(M \times I, F \circ (g \times 1_I))$. Of course, it turns out that the other properties of a homology theory are also satisfied.

In fact, because $\delta(M \times I) = M \amalg M$, $MO_*(X)$ is a vector space over $\mathbb{Z}/2$. We can impose extra struture on all manifolds in sight to get variants of this homology theory. Working with oriented manifolds leads to groups $MSO_*(X)$, which are not killed by multiplication by 2.

We get an even more interesting theory by considering suitable complex manifolds. A manifold is called *stably complex* if it admits a complex linear structure in its stable normal bundle, i.e. the normal bundle obtained by embedding in a large dimensional Euclidean space.

Definition 1.4.4. The set of bordism classes of stably complex manifolds, $MU_n(X)$, is a group under disjoint union, the *n-th complex bordism group* of X.

Again we can make $MU_*(-)$ into a homotopy functor and all the axioms of a homology theory are satisfied.

The coefficient groups MU_* are obtained by taking X to be a point, and, since there is only one map to a point, they consist of bordism classes of manifolds. In fact MU_* is not just a graded abelian group; it is a graded ring with the product given by Cartesian product of manifolds. An element of MU_* is the bordism class of a stably complex manifold, $[M]$, and we have a product structure coming from

$$MU_* \otimes MU_* \to MU_*, \qquad [M] \otimes [N] \mapsto [M \times N].$$

Theorem 1.4.5 (Milnor, Novikov)**.** *$MU_* = \mathbb{Z}[x_1, x_2, x_3, \dots]$, where $|x_i| = 2i$.*

There are no canonical generators here, but explicit descriptions can be given for generators x_i.

We have mentioned several versions of bordism; there are others obtained by imposing different structures on all the manifolds in sight.

(4) **Morava K-theories.** For each prime p (often omitted from the notation) and for $n \geqslant 0$, there is a homology theory $K(p, n)_*(-)$, the *n-th Morava K-theory.* For $n = 0$ and $n = 1$ the theories are reasonably familiar: $K(p, 0)_*(-)$ is ordinary homology with rational coefficients; $K(p, 1)_*(-)$ is

very closely related to complex K-theory "mod p". (We will return to theories with coefficients soon.) The values on a point are given by

$$K(p,0)_* = \mathbb{Q},$$
$$K(p,n)_* = \mathbb{F}_p[v_n, v_n^{-1}], \qquad \text{for } n \geqslant 1,$$

where $|v_n| = 2p^n - 2$.

We will not explain how to construct the Morava K-theories here, but we mention that it may be done in a similar manner to bordism, using certain manifolds with singularities.

(5) **Elliptic theories and TMF.** There are lots of variants of elliptic homology, $Ell_*(-)$. They all share the property that Ell_* is some ring of modular forms. These theories are supposed to have a strong geometric flavour, related to elliptic curves. However, at present there is no geometric construction. We will return to the construction of one such theory in the next section. Putting all elliptic theories together produces a theory known as $TMF_*(-)$, standing for "topological modular forms". (Slightly less vaguely, TMF is a homotopy inverse limit over all elliptic theories.) This is a periodic theory, with period 576. There is a natural transformation $TMF_*(-) \to Ell_*(-)$ for any elliptic theory $Ell_*(-)$.

(6) **Stable homotopy.** Let

$$\pi_*^S(X) = \lim_{n\to\infty} \pi_{n+*}(S^n X),$$

the stable homotopy groups of X. This is a (reduced) homology theory, the most complicated of all of them. In particular, the coefficient groups for the theory, given by the values on S^0, are the stable homotopy groups of the spheres. These are not known, but are known to be extremely complicated. Here are the first few groups.

n	0	1	2	3	4	5	6	7	8
π_n^S	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
	degree	Hopf							

2. Theories with extra structure : products and operations

2.1. **Why products? Why operations?** It turns out that all the cohomology theories we have mentioned have more structure than simply that of a graded abelian group. In general terms, we would like to endow our cohomology theories with as much structure as possible. The richer the target category, the more chance we have of using our functor to distinguish objects. Of course, for it to be useful, we have to be able to calculate with the extra structure.

Ordinary cohomology has products; the *cup product* makes $H^*(X)$ into a graded commutative ring. This ring structure is usually calculable and it is certainly useful. One can find examples of two different spaces whose cohomology is the same as graded abelian groups but different as graded rings. We can show, for example, that $\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to ΣX for any X, since $H^*(\mathbb{C}\mathrm{P}^2)$ has non-trivial products, but cup-products are trivial in the cohomology of a suspension.

Similar remarks apply to operations. Roughly, an operation in a cohomology theory $E^*(-)$ is a map $E^*(X) \to E^*(X)$, natural in X, which may or may not preserve various structures on $E^*(X)$. One can find examples of two different spaces with the same graded cohomology rings, but distinguished by different actions of certain natural operations on those rings. (See problems.)

Another use for operations comes in splitting cohomology theories into smaller simpler parts. One way to prove the existence of such splittings is to find idempotent operations.

In many cases the most useful structures occur in theories "with coefficients", so we first say something about this.

2.2. **Introducing coefficients.** For many purposes, we want to be able to work "one prime at a time". This means introducing coefficients such as the prime field $\mathbb{F}_p$, the p-local integers

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : (p, b) = 1 \right\}$$

or the p-adic integers $\mathbb{Z}_p$. (The ring of p-adic integers $\mathbb{Z}_p$ is a ring obtained by completing $\mathbb{Z}$ with respect to a norm defined in terms of divisibility by the prime p. Alternatively, one can describe it as the inverse limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$.) Usually the coefficients will be a ring.

Given a homology theory $E_*(-)$, and a ring R, there is another homology theory $ER_*(-)$, *E-homology with coefficients in R*. We may also write $E_*(X; R)$ for $ER_*(X)$. Similar remarks apply to E-cohomology with coefficients.

For example, coefficients are introduced into ordinary homology by replacing the chain complex $C_*(X)$ by $C_*(X) \otimes R$. There is a Universal Coefficient Theorem relating $H_*(X; R)$ to $H_*(X)$. For a general theory E this gets replaced by a Universal Coefficient Spectral Sequence relating $ER_*(-)$ and $E_*(-)$.

Similarly, we introduce coefficients R into ordinary cohomology by replacing the cochain complex $\mathrm{Hom}(C_*(X), \mathbb{Z})$ by $\mathrm{Hom}(C_*(X), R)$. Alternatively, we can characterize ordinary cohomology with coefficients in R by the axioms 1.3.1 together with the modified dimension axiom

$$H^n(pt; R) = \begin{cases} R, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

The general procedure for introducing coefficients is most easily described at the spectrum level, using Moore spectra; we leave this to JH's course.

Important examples of theories with coefficients include mod p cohomology and p-adic K-theory.

2.3. **Products.** To say that a cohomology theory *has products* (or is a *ring theory* or is a *multiplicative theory*) means that for any space X, there is a product map

$$E^*(X) \otimes E^*(X) \to E^*(X),$$

making $E^*(X)$ into a graded ring. In almost all the examples this will be a graded commutative ring. Of course, we really mean that the functor $E^*(-)$ takes values in the category of graded (commutative) rings, so in addition to the above, for $f : X \to Y$, the induced map $f^* : E^*(Y) \to E^*(X)$ is a ring homomorphism.

In particular, we may take X to be a point and then the coefficients $E^* = E_{-*}$ of our theory will be a graded commutative ring.

Ordinary cohomology. As mentioned above, in ordinary cohomology we have the *cup-product*. For singular cohomology, it can be defined as follows. Let $f \in \mathrm{Hom}(C_m(X), \mathbb{Z})$ and $g \in \mathrm{Hom}(C_n(X), \mathbb{Z})$. Then define $f \cup g \in \mathrm{Hom}(C_{m+n}(X), \mathbb{Z})$ by specifying the value on a singular simplex $\sigma : \Delta^{m+n} \to X$ to be

$$(f \cup g)(\sigma) = f(\sigma|_{[v_0,\ldots,v_m]})g(\sigma|_{[v_m,\ldots,v_{m+n}]}).$$

This passes to a well-defined product on cohomology classes, making $H^*(X)$ into a graded commutative ring. The same construction works for cohomology with coefficients in any ring R.

Application. Knowledge of the ring structure of $H^*(\mathbb{R}\mathrm{P}^{\mathrm{n}} \times \mathbb{R}\mathrm{P}^{\mathrm{n}}; \mathbb{Z}/2)$ is already enough to prove that if $\mathbb{R}^n$ is a division algebra, then n is a power of 2.

K-theory. Here the product has a natural geometric description in terms of vector bundles. Recall that the tensor product $V_1 \otimes V_2$ of two complex vector spaces V_1 and V_2 is again a complex vector space. If V_1 and V_2 have dimensions m and n respectively then $V_1 \otimes V_2$ has dimension mn. This operation can be extended to vector bundles. Given vector bundles E and F over spaces X and Y respectively, there is a bundle $E \otimes F$ over $X \times Y$ such that the fibre over (x, y) is $E_x \otimes F_y$. The construction passes to isomorphism classes of vector bundles and to formal differences of such classes, giving

$$K^0(X) \otimes K^0(Y) \to K^0(X \times Y).$$

Then we can let $Y = X$ and compose with $\Delta^* : K^0(X \times X) \to K^0(X)$, where $\Delta : X \to X \times X$ is the diagonal map, giving our product map

$$K^0(X) \otimes K^0(X) \to K^0(X).$$

The product in a theory can also be described at the level of the representing spectrum, making it a *ring spectrum.* Further details will be given in John Hunton's course.

In particular, the complex K-theory spectrum is a ring spectrum and so we get a graded ring structure on the K-theory of any space X:

$$K^*(X) \otimes K^*(X) \to K^*(X).$$

Cobordism. Again the cohomology theory $MU^*(-)$ has a product. We have already seen this in the coefficient groups $MU^{-*} = MU_*$, coming from Cartesian product of manifolds. Of course, the geometric theory is *homology* not cohomology. To see the product, you really have to go to the representing spectrum MU. It turns out to be easy to see that this *is* a ring spectrum, giving a product on $MU^*(X)$ for any space X.

Stable cohomotopy. Again here the more natural theory is the homology theory, but there is a ring structure in the associated cohomology theory, stable cohomotopy, denoted $\pi_S^*(-)$, with

$$\pi_S^*(X) = \lim_{n\to\infty}[S^n X, S^{n+*}].$$

This is easily seen to be a graded ring, using the diagonal map on spaces and smash products of maps.

Other theories. All the theories we have mentioned have products. The Morava K-theories $K(p,n)^*(-)$ have a product, commutative for $p \neq 2$. The various elliptic theories have products and so does TMF.

2.4. Operations.

Definition 2.4.1. For a cohomology theory $E^*(-)$, a *cohomology operation of type (p,q)* is a natural transformation $\theta : E^p(-) \to E^q(-)$ of functors regarded as taking values in sets. The operation θ is *additive* if it is a natural transformation of functors regarded as taking values in groups.

A *stable cohomology operation of degree* q is a sequence of cohomology operations $\theta^n : \tilde{E}^n(-) \to \tilde{E}^{n+q}(-)$, commuting with the suspension isomorphisms, i.e. such that $\theta^n \circ \sigma = \sigma \circ \theta^{n+1}$ for all $n \in \mathbb{Z}$. One often writes θ for the sequence $\{\theta^n\}$.

We write $A(E)^q$ for the set of all stable cohomology operations of degree q for a cohomology theory $E^*(-)$. We can make this into an abelian group by setting $(\theta + \varphi)(x) = \theta(x) + \varphi(x)$ for all $x \in E^*(X)$. Composition of operations defines a pairing

$$A(E)^p \times A(E)^q \to A(E)^{p+q}.$$

It turns out that the component maps of a stable operation are always additive (see problems) and so composition is bilinear and passes to a pairing

$$A(E)^p \otimes A(E)^q \to A(E)^{p+q}.$$

This makes $A(E)^* = \oplus_q A(E)^q$ into a graded ring. In fact it is a graded E^*-algebra, as for $r \in E^*$ and $\theta \in A(E)^*$, we can define $(r\theta)(x) = r(\theta(x))$.

For any X, $E^*(X)$ is a graded module over the graded ring of operations $A(E)^*$.

Examples.

Ordinary cohomology. The *Steenrod algebra* $A(p)^*$ is defined to be the algebra of operations for ordinary cohomology with $\mathbb{F}_p$ coefficients. So $H^*(X;\mathbb{F}_p)$ is a module over this algebra for any X. The algebra is complicated, but can be described by explicit generators and relations. This gives an extremely rich and useful extra structure to ordinary cohomology.

This will be discussed in more detail in Martin Crossley's talk (particularly concentrating on $p = 2$ and Steenrod's operations Sq^{i}).

K-theory. In K-theory, there are very nice examples of operations one can see geometrically, namely *exterior powers*. The standard constructions of symmetric and exterior powers on vector spaces pass easily to vector bundles and to isomorphism classes of vector bundles. To define them for K-theory involves working out what they should be on formal differences and it turns out that we can define the k-th exterior power operation by $\lambda^k(\xi - \eta) = \sum_{i+j=k}(-1)^j \Lambda^i(\xi) S^j(\eta)$. This construction is natural with respect to pull-backs and so it gives a (non-additive) unstable degree zero operation.

There is a neat trick to get from exterior powers to operations with much nicer properties, the *Adams operations.* For $x \in K^0(X)$, write $\Lambda_t(x)$ for the formal power series $\sum_{k=0}^{\infty} \Lambda^k(x)t^k \in K^0(X)[[t]]$. Define a new power series $\Psi_t(x) = \mathrm{dim(x)} - \mathrm{t}\frac{\mathrm{d}}{\mathrm{dt}} \log \Lambda_{-\mathrm{t}}(\mathrm{x})$ and let $\Psi^k(x)$ be the coefficient of t^k in this series. This gives unstable operations $\Psi^k : K^0(-) \to K^0(-)$ for $k \in \mathbb{N}$.

Theorem 2.4.2 (Adams)**.** *There are unstable operations $\Psi^k : K^0(-) \to K^0(-)$ for $k \in \mathbb{Z}$ with the following properties.*

(1) *Ψ^k is a ring homomorphism $K^0(-) \to K^0(-)$.*
(2) *$\Psi^k \Psi^l = \Psi^{kl}$.*
(3) *If ξ is a line bundle, then $\Psi^k(\xi) = \xi^k$.*
(4) *If p is a prime, then $\Psi^p(x) = x^p \mod p$.*
(5) *On $\tilde{K}^0(S^{2n}) = \mathbb{Z}$, Ψ^k acts as multiplication by k^n.*

Application. Famously, unstable Adams operations in K-theory can be used to give a nice proof, due to Adams and Atiyah, of the following.

Theorem 2.4.3. *$\mathbb{R}^n$ is a division algebra if and only if $n = 1, 2, 4$ or 8.*

The four division algebras are $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. The same result had previously been proved by a more involved method involving secondary operations in ordinary cohomology.

If one tries to extend the operation Ψ^k to a degree zero stable operation, one finds one needs to divide by k in order to have component maps Ψ^k/k^n. Thus only Ψ^1 (the identity operation) and Ψ^{-1} (complex conjugation) give rise to stable operations in $A(K)^*$. On the other hand, the algebra $A(K)^*$ is known to be uncountable. It is an open problem to describe some more elements in it.

At the price of introducing coefficients, one can make more progress. For example, with $\mathbb{Z}_p$ coefficients, each Adams operation Ψ^α with $\alpha \in \mathbb{Z}_p^\times$ gives a stable operation. This allows one to give a complete description of $A(K\mathbb{Z}_p)^*$. Recently, the same has been done for $A(K\mathbb{Z}_{(p)})^*$.

Cobordism. The ring of stable complex cobordism operations $A(MU)^*$ was determined by Landweber and Novikov. It is large and complicated. In particular, for each finite non-decreasing sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots)$ there is a *Landweber-Novikov operation*, s_α, of degree $2|\alpha| = 2\sum_i \alpha_i$.

2.5. **Cooperations.** So far discussion in this section has concentrated on cohomology, where we have considered products and operations. One could ask about extra structure in homology too. It turns out that there are some technical reasons why this is a rather good idea.

In terms of the spectrum E representing a cohomology theory, the operations are given by homotopy classes of maps $E \to E$, that is, the E-cohomology of E itself: $A(E)^* = E^*(E) = [E, E]_*$. If one wants to work in homology then one considers the so-called *cooperations*, that is the E-homology of E itself, $E_*(E)$.

For nice theories with products, the cooperations $E_*(E)$ is a *Hopf algebroid.* This means that it has lots of structure, all of which comes from spectrum level maps. John Hunton will say more about this. In particular, $E_*(E)$ is a ring and it is a bimodule over the coefficients E_*. It is also a coalgebra, with comultiplication map dual to the composition of operations. Where $E^*(X)$ is a module over the ring of operations for any X, in homology $E_*(X)$ is a *comodule* over the cooperations.

In practice much work is done in the homological setting at the price of having to work with comodules and coalgebras rather than modules and algebras. The main technical reason for this is that in homology we get *discrete* rings and modules, whereas in cohomology, rings and modules come with a natural topology which one should take into account and which can present conceptual and technical difficulties.

Generally the cooperations are easier to calculate, as explicit elements can be obtained from the E-homology of spaces. In favourable cases one can then obtain operations as the dual of cooperations.

Examples.
(1) **Ordinary cohomology.** Again we work with $\mathbb{F}_p$ coefficients. The cooperation ring is Milnor's dual of the Steenrod algebra. In particular, for $p = 2$, it is a polynomial ring

$$\mathbb{F}_2[\xi_1, \xi_2, \dots],$$

where $|\xi_n| = 2^n - 1$. Here $\xi_1^{2^i}$ is dual to the Steenrod operation Sq^{2^i} and ξ_n is dual to an operation denoted Q_{n-1}.
(2) **Bordism**. Here the structure of the cooperations was determined independently by Landweber and Novikov. We have

$$MU_*(MU) = MU_*[b_1, b_2, \dots]$$

where $|b_i| = 2i$, and this is sometimes called the *Landweber-Novikov* algebra. Here b_i "comes from" the generator $b_i \in H_{2i}(\mathbb{C}\mathrm{P}^\infty)$. The Landweber-Novikov operation s_α is dual to the monomial $b_1^{\alpha_1} \dots b_r^{\alpha_r}$.

The Adams-Novikov spectral sequence is set up in this context of cooperations. It is a powerful tool for calculating homotopy classes of maps. The E_2-term is $\mathrm{Ext}_{E_*(E)}(E_*(X), E_*(Y))$, this being Ext of comodules over the coalgebra $E_*(E)$. In favourable cases, it converges, roughly speaking, to the homotopy classes of maps from X to Y that are visible to the theory $E_*(-)$. This spectral sequence will be discussed in John McCleary's lectures.

3. New cohomology theories from old

3.1. **Splittings.** It sometimes happens that introduction of coefficients simplifies things. Suppose that, for a cohomology theory $E^*(-)$, we have an idempotent stable cohomology operation, e. Thus $e \in A(E)^*$ and $e^2 = e$. Then, for any X, the $A(E)^*$-module $E^*(X)$ splits as $E^*(X) = eE^*(X) \oplus (1-e)E^*(X)$. The functors $eE^*(-)$ and $(1-e)E^*(-)$ are cohomology theories and so the cohomology theory $E^*(-)$ splits into smaller pieces $eE^*(-)$ and $(1-e)E^*(-)$.

We note two important examples.

(1) **The Adams summand of K-theory.** There is a cohomology theory, denoted G^* (or sometimes L^*) and called the *Adams summand* of p-local K-theory (for p odd), such that

$$K^*(X;\mathbb{Z}_{(p)}) \cong \bigoplus_{i=0}^{p-2} G^{*+2i}(X)$$

for all spaces X. This means that $K^*(-;\mathbb{Z}_{(p)})$ is completely determined by $G^*(-)$ and we may as well work with the simpler theory $G^*(-)$. This is an example of a *finite* splitting, into $p-1$ parts.

Adams proved this by showing that there are idempotent cohomology operations $e_0,\dots,e_{p-2}$ in $A(K\mathbb{Z}_{(p)})^*$ such that $e_0+\cdots+e_{p-2}=1$. Then $G^*(-) = e_0 K\mathbb{Z}^*_{(p)}(-)$ and it turns out that the other summands in the splitting are just shifted copies of $G^*(-)$. Products pass naturally to the e_0 summand, so that $G^*(-)$ is a theory with products. We have $G^* = \mathbb{Z}_{(p)}[t^{p-1},t^{-(p-1)}]$.

(2) **Brown-Peterson theory,** BP. A similar thing happens with p-local cobordism. Quillen showed that there is an idempotent cohomology operation $e \in A(MU\mathbb{Z}_{(p)})^*$ such that e acts on the coefficient groups $MU_* = \mathbb{Z}[x_1,x_2,\dots]$ as the ring homomorphism determined by

$$x_i \mapsto \begin{cases} x_i & \text{if } i = p^k-1 \text{ for some k,} \\ 0 & \text{otherwise.}\end{cases}$$

It follows that there is a homology theory, called $BP_*(-)$, with $BP_* = \mathbb{Z}_{(p)}[v_1,v_2,\dots]$, $|v_i| = 2p^i-2$. It turns out that

$$MU_*(X;\mathbb{Z}_{(p)}) = \bigoplus_{i=0}^{\infty} BP_{*+n(i)}(X).$$

Here n is some function of i such that $n(i)\to\infty$ as $i\to\infty$.

This time we have infinitely many pieces in the splitting and again $MU_*(-;\mathbb{Z}_{(p)})$ is determined by the much more manageable theory $BP_*(-)$.

3.2. **Localization.** Let $E_*(-)$ be a homology theory.

Definition 3.2.1. A space Y is *E_*-local* if whenever a map $f: X_1 \to X_2$ is such that $E_*(f): E_*(X_1)\to E_*(X_2)$ is an isomorphism then the map $f^*: [X_2,Y]\to[X_1,Y]$ is also an isomorphism.

Definition 3.2.2. An *E_*-localization* of a space or spectrum X is a map η_X from X to an E_*-local space or spectrum (usually denoted by $L_E X$), such that

(1) $E_*(\eta_X)$ is an isomorphism, and
(2) for any map $f: X\to Y$ inducing $E_*(X)\cong E_*(Y)$, there is a map $r: Y\to L_E X$ (unique up to homotopy) with $rf=\eta_X$.

Theorem 3.2.3 (Bousfield). *The localization $\eta_X : X \to L_E X$ exists for any homology theory E and any space or spectrum X.*

A more intuitive way of thinking about localization is to consider what information is visible to the theory $E_*(-)$. Tautologically, if $E_*(X) = 0$

then $E_*(-)$ cannot detect X. So it makes sense to pass from our homotopy category of spaces or spectra to the quotient by $ann(E) = \{X \mid E_*(X) = 0\}$. This quotient category is equivalent to the category of E-local objects and we can think of the localization $L_E X$ of X as corresponding to a choice of representative of X in the quotient category.

In particular, as we have seen that spectra give cohomology theories, we can use Bousfield localization for spectra to produce new examples.

Localizations with respect to connective theories (those represented by spectra with homotopy groups trivial below some fixed degree) are quite well understood. For example, the introduction of coefficients $\mathbb{Z}_{(p)}$ can be done by a Bousfield localization (with respect to the corresponding Moore spectrum $M = M\mathbb{Z}_{(p)}$; we have $L_M X \simeq X \wedge M$). We call this localization the p-localization of X and may write it $X_{(p)}$. Other Bousfield localizations are complicated and mysterious.

Perhaps more importantly, Bousfield localization provides a systematic way of organizing information from homology theories. This is the chromatic viewpoint, which we will return to later.

3.3. **Formal group laws and Landweber exact theories.** Shortly we will see how to produce homology theories from certain MU_*-modules. Formal group laws (FGLs) are closely related to such modules.

Definition 3.3.1. Let R be a commutative ring with 1. An *FGL over R* is a power series $F(x, y)$ over R such that

(1) (identity) $F(x, 0) = F(0, x) = x$,
(2) (commutativity) $F(x, y) = F(y, x)$,
(3) (associativity) $F(F(x, y), z)) = F(x, F(y, z))$.

Examples.
(1) $F(x, y) = x + y$, the additive FGL.
(2) $F(x, y) = x + y + xy$, the multiplicative FGL.
(3) $F(x, y)$ is the power series expansion of $\frac{x\sqrt{E(y)}+y\sqrt{E(x)}}{1-\varepsilon x^2y^2}$, where $E(x) = 1 - 2\delta x^2 - \varepsilon x^4$. This is a FGL over $R = \mathbb{Z}[1/2][\delta, \varepsilon]$. It is associated with the elliptic curve $y^2 = E(x)$ and so is an example of an *elliptic FGL*.

The relationship between FGLs and bordism is provided by important theorems of Lazard and Quillen.

Theorem 3.3.2 (Lazard)**.** *(1) There is a universal FGL defined over a ring L, of the form*

$$G(x, y) = \sum_{i,j} a_{i,j} x^i y^j, \qquad a_{i,j} \in L,$$

such that, for any FGL F over R, there is a unique ring homomorphism $\theta : L \to R$ with $F(x, y) = \sum_{i,j} \theta(a_{i,j}) x^i y^j$.
(2) $L = \mathbb{Z}[x_1, x_2, \dots]$, where $|x_i| = -2i$ and $|a_{i,j}| = 2(1 - i - j)$.

We have noted that $MU^*(-)$ has products, making $MU^*(X)$ into a graded MU_*-algebra. We have $MU^*(\mathbb{C}\mathrm{P}^\infty) = MU^*[[x]]$ where $|x| = 2$ and $MU^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong MU^*[[x \otimes 1, 1 \otimes x]]$. As $\mathbb{C}\mathrm{P}^\infty$ is a topological group, we have a product map $\mu : \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ inducing

$$\begin{aligned} MU^*(\mathbb{C}\mathrm{P}^\infty) &\to MU^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \\ x &\mapsto \mu^*(x) = F(x \otimes 1, 1 \otimes x). \end{aligned}$$

Then F is a FGL over MU^* and by Lazard's Theorem corresponds to a ring homomorphism $\theta : L \to MU^*$.

Theorem 3.3.3 (Quillen)**.** *The ring homomorphism $\theta : L \to MU^*$ is an isomorphism.*

So the MU FGL is the universal FGL. We identify L and MU^* by the isomorphism θ. The conclusion is that any FGL over R gives a ring homomorphism $MU^* \to R$.

Now suppose that M is a module over $MU_* = \mathbb{Z}[x_1, x_2, \dots]$. We could consider the functor $\mathcal{CW} \to \mathcal{A}_*$ given by $X \mapsto MU_*(X) \otimes_{MU_*} M$.

The Landweber exact functor theorem (LEFT) tells us when this functor is a homology theory.

Theorem 3.3.4 (Landweber)**.** *The functor $X \mapsto MU_*(X) \otimes_{MU_*} M$ is a homology theory if and only if for every prime p, multiplication by p on M and by x_{p^n-1} on $M/I(p,n)M$ for $n > 0$ is injective. Here $I(p,n) = (p, x_{p-1}, x_{p^2-1}, \dots, x_{p^{n-1}-1})$, a prime ideal in MU_*.*

So this gives us a method for building new theories out of cobordism from algebraic data.

Examples.

(1) **Ordinary cohomology.** The same process with $\mathbb{C}\mathrm{P}^\infty$ that gave the MU FGL, gives an FGL associated with ordinary cohomology. This is just the additive FGL. If we use rational coefficients, the conditions of LEFT are satisfied and we get $H_*(X;\mathbb{Q}) \cong MU_*(X) \otimes_{MU_*} \mathbb{Q}$.

(2) **K-theory.** Again we can do the same thing for K-theory; we get the multiplicative FGL. We can recover K-homology from bordism this way. Associated to the multiplicative FGL, there is a ring homomorphism called the Todd genus, $MU_* \to K_*$, making K_* into a module over MU_* and the conditions of LEFT are satisfied. So $K_*(X) \cong MU_*(X) \otimes_{MU_*} K_*$, recovering a result proved earlier by Conner and Floyd.

(3) **Brown-Peterson theory.** This is one way to make $BP_*(-)$, since $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ is an MU_*-module via the ring homomorphism

$$\begin{aligned} \mathbb{Z}[x_1, x_2, \dots] &\to \mathbb{Z}_{(p)}[v_1, v_2, \dots] \\ x_i &\mapsto \begin{cases} v_n, & \text{if } i = p^n - 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The conditions of LEFT are satisfied.

(4) **Some versions of elliptic homology.** One version of elliptic homology has $Ell_* = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon, \Delta^{-1}]$, where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$. The grading is given by $|\delta| = 2$ and $|\varepsilon| = 4$. We have defined a FGL over Ell_* and hence there is a ring homomorphism $\varphi_{Ell} : MU_* \to Ell_*$. This makes Ell_* into an MU_*-module satisfying the conditions of LEFT and so we can define our elliptic

homology theory by

$$Ell_*(X) = MU_*(X) \otimes_{MU_*} Ell_*.$$

This gives a periodic theory with period 12.

(5) **Johnson-Wilson theories.** Let $E(p,n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$. This is an MU_*-module satisfying the conditions and so there is a homology theory $E(p,n)_*(-)$ with these coefficients, the n-th Johnson-Wilson theory.

We see that cobordism is a theory containing lots of information. The LEFT and FGL methods allow us to produce other interesting theories from cobordism. They will contain less information, but typically be more calculable. Not all theories come this way. The Morava K-theories *cannot* be obtained in this manner; neither can TMF.

4. Relationships between cohomology theories

4.1. **Extracting information from cohomology theories.** The main tool for calculations is the spectral sequence, as explained in John McCleary's lectures. In particular, we have the Atiyah-Hirzebruch spectral sequence and the Adams spectral sequence. For a homology theory $E_*(-)$, the former takes as input the ordinary homology of a CW-complex X with coefficients in E_* and converges to $E_*(X)$. The later allows us to recover information about homotopy classes of maps from the values of a theory $E_*(-)$.

4.2. **How much information is in a theory?** There is a trade-off between computability and information content. Ultimately we would like to know about stable homotopy, but this is very difficult to compute. We have already noted that the stable homotopy groups of the spheres are not known. Indeed we do not know the stable homotopy groups of any finite complex, except finite $K(\pi, 1)$s.

The bordism groups $MU_*(X)$ provide a very powerful invariant of X. While complex cobordism is not quite so powerful as stable homotopy itself, one can obtain stable homotopy as a variant called *stably framed bordism*. Generally $MU_*(X)$ is very large and it is sometimes hard to compute.

4.3. **Maps between theories.** We have now seen several examples of (co)homology theories and we have mentioned ways to produce many more. We would like to understand the relationships between them. One way to do this is to give maps from one theory to another.

We will just mention some important examples.

(1) **Chern character.** The Chern character provides a way of comparing K-theory and ordinary cohomology with rational coefficients. We have ring homomorphisms $\mathrm{ch} : K^0(X) \to H^{\mathrm{ev}}(X;\mathbb{Q})$ and $\mathrm{ch} : K^1(X) \to H^{\mathrm{odd}}(X;\mathbb{Q})$ such that $\mathrm{ch} : (K^0(X) \oplus K^1(X)) \otimes \mathbb{Q} \cong H^*(X;\mathbb{Q})$.

(2) **Maps between different flavours of a theory.** There is a map $K^0(-) \to KO^0(-)$ given by forgetting the complex structure of a complex vector bundle. Similar remarks apply to flavours of cobordism.

(3) **Landweber exact theories.** We have seen how to produce new theories $E_*(-)$ out of $MU_*(-)$ from algebraic data. The construction gives us a natural transformation $MU_*(-) \to E_*(-)$.

(4) **Splittings.** A splitting exhibits one theory as a direct summand of another, so of course, in this situation, there are natural transformations in both directions between the theories.

(5) **Localizations.** A Bousfield localization $F \to L_E F$ gives a map of homology theories $F_*(-) \to L_E F_*(-)$.

4.4. **The chromatic viewpoint.** A more systematic approach to organizing the information provided by different homology theories is given by the *chromatic filtration*. Here we just sketch the idea, due to Ravenel.

Stable homotopy is the most complicated homology theory and the one we would ultimately like to understand. Stable homotopy theory (localized at a prime) is naturally filtered by *chromatic layers*. Ordinary cohomology sees only chromatic layer 0; K-theory sees layers 0 and 1; elliptic cohomology sees layers 0, 1 and 2. Higher layers are detected by the Morava K-theories $K(p, n)$.

Using the LEFT we constructed a homology theory $E(p, n)_*(-)$ with $E(p, n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$. These give (for each prime p) a hierarchy of homology theories $E(p, n)_*(-)$, getting more and more complicated as n gets larger. The Morava K-theory, $K(p, n)_*(-)$, measures the difference between level $n-1$ and level n in this tower of homology theories. (Indeed, for our purposes here, we could use $\bigvee_{i=0}^{n} K(p, i)$ in place of $E(p, n)$. They have the same "Bousfield class".)

The thing that makes this approach powerful is that the Morava theories are highly computable, principally because we have a Kunneth Theorem.

Theorem 4.4.1. *For all primes p and $n \geqslant 0$, if $E_*(-) = K(p, n)_*(-)$, then for all X and Y,*

$$E_*(X \wedge Y) = E_*(X) \otimes_{E_*} E_*(Y).$$

Any other theory $E_*(-)$ with this property is determined by the Morava K-theories and $H_*(-; k)$ where k is a field.

The localization functors L_n corresponding to the theories $E(p, n)_*(-)$ are also increasingly complicated. The chromatic convergence theorem tells us that we can reassemble the homotopy type of a suitable X from its localizations with respect to these theories.

Definition 4.4.2. The *chromatic tower* of X is the inverse system

$$L_0 X \leftarrow L_1 X \leftarrow L_2 X \leftarrow \cdots ,$$

where L_n is localization with respect to $E(p, n)_*(-)$.

Theorem 4.4.3 (Hopkins-Ravenel)**.** *For a finite CW-complex X, the chromatic tower converges to the p-localization of X:*

$$X_{(p)} \simeq \varprojlim L_n X.$$

5. More structure and an open problem

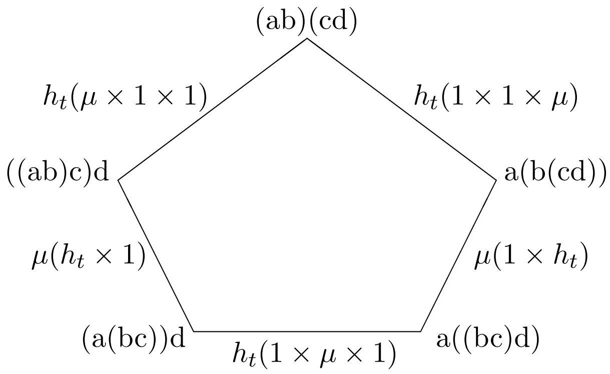
5.1. **Why yet more structure?** We have already discussed products and operations in cohomology theories. However, for some purposes, one would like even richer structure. We will just sketch the beginnings of this topic and mention some of the results.

We have said that the product in a theory comes from a spectrum level product. This really lives in a stable *homotopy* category, so it is supposed to be associative, and have a unit, (and possibly be commutative), *up to homotopy.*

Let us explore the idea of a product *associative up to homotopy* in the more familiar context of spaces. Suppose we have a space X with a map $\mu : X \times X \to X$; such a space is called an A_2 *space.* Now suppose that

$$\mu(\mu \times 1) \simeq \mu(1 \times \mu).$$

A loop space is an example of such a space, of course. Let $h_t : X \times X \times X \to X$ be a homotopy between $\mu(\mu \times 1)$ and $\mu(1 \times \mu)$. A space X with a product and an explicit choice of such a homotopy is called an A_3 *space.* We write xy for $\mu(x, y)$. There are five possible maps $X^4 \to X$ corresponding to different orders of multiplication and also homotopies between them:



We can put these five maps together to define a map $S^1 \times X^4 \to X$ and then we can ask whether this maps extends to a map $D^2 \times X^4 \to X$. If it does, the previous structure together with an explicit choice of such an extension makes X an A_4 *space.* Stasheff showed how to continue this pattern, to encode all the higher homotopy conditions arising, thus defining an A_n *space.* A space which is an A_n space for all n is called an A_∞ *space.* It is a space with a product associative up to all higher homotopies.

Brushing plenty of technical trouble under the carpet, we can carry out a similar process which builds in all higher homotopies for commutativity as well as associativity, leading to the notion of an E_∞ space. And we can do a similar thing for spectra, arriving at A_∞ and E_∞ ring spectra. These structures were studied by Boardman and Vogt and by May and his coworkers in the 1970s.

Nowadays there is a good category of spectra whose homotopy category is the stable homotopy category. (For a surprisingly long time no such category was known and its existence was doubted. The history of the subject has been convoluted and long.) Actually there are now lots of different descriptions. In one formulation, the objects are *S-modules.* The analogues of

A_∞ ring spectra are S-algebras and the analogues of E_∞ ring spectra are commutative S-algebras.

There are good reasons to be interested in these kinds of extra structure.
(1) The extra structure at the spectrum level does give rise to extra structure in cohomology, for example secondary or higher operations.
(2) New theories may be constructed exploiting this kind of extra structure. For example, the construction of TMF uses methods involving highly structured ring spectra.

5.2. **Examples.** The spectra representing ordinary cohomology (with coefficients in a commutative ring), connective versions of K-theory and various cobordism theories are all commutative S-algebras. These cases were accessible to the methods developed by Boardman and Vogt and by May and coworkers.

In the new categories one can show that Bousfield localization preserves commutative S-algebras, so we can produce other examples, including periodic K-theory.

5.3. **Machinery and harder examples.** Considerable effort has been put into developing tools for deciding whether a given theory has all this desirable extra structure. There are several approaches, but they share basic characteristics. All associate to a theory elements lying in some kind of algebraic cohomology groups. These elements are obstructions to the existence of the required structure: if the elements vanish, then the structure exists. In practice, as so often with obstruction theory, the only known way to show the particular elements you are interested in vanish is to show that the whole group is zero. The input data for these theories is basically the structure of the cooperations for the theory.

For A_∞ structures, the theory goes back to the 1980's, due to Alan Robinson and further developed by Andy Baker. For E_∞ structures, it is a bit more recent, starting in the 1990's and continuing to the present. One approach, known as Γ-cohomology, was developed by Alan Robinson, with some input from myself. Another approach is due to Maria Basterra. And there is a whole programme of work by Paul Goerss and Mike Hopkins in this direction.

These methods have been exploited recently to prove that various theories are highly structured. For example, Baker and Richter have shown that the Adams summand is a commutative S-algebra. And, Goerss and Hopkins, following work of Hopkins and Miller, have shown that the so-called Morava E-theories are represented by commutative S-algebras.

5.4. **An open problem.** However the answer to the following question is still not known. Is BP a commutative S-algebra?

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