Elliptic Cohomology

Prospects in Mathematics
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Plan

Overview

What is elliptic cohomology?

- It's a cohomology theory.
  For each topological space X, we have a graded ring $Ell^*(X)$, the elliptic cohomology of X.
- There are various different versions.
  In some sense there's one version $Ell^*_C(-)$ for each elliptic curve C, hence the name.
  This provides a strong connection to number theory.
- It's also related to theoretical physics - string theory and conformal field theory.
- The current definition is via homotopy theory.
- It's a very active research area, especially the search for a more geometric definition.

A very brief history

- 1980's: Witten: invariants of manifolds related to string theory, "physical" proof of a mysterious connection with elliptic curves
- 1980's, 1990's: Ochanine, Landweber, Stong, Ravenel: elliptic genus and first versions of elliptic cohomology
- 1980's, 1990's: Segal: "elliptic objects" - relation with conformal field theory
- Early 2000's: Ando, Hopkins, Strickland: good homotopy theory definition of all elliptic theories
- Early 2000's: Hopkins, Lurie, Miller: homotopy theory construction of the "universal elliptic cohomology", $tmf^*(-)$
- Now: ongoing search for a geometric or analytical or physical description

Invariants

In algebraic topology we assign invariants to topological spaces X.

$$\text{topological spaces} \rightsquigarrow \text{algebraic gadgets of some kind}$$

Examples

- The fundamental group

$$X \mapsto \pi_1(X).$$

- The Euler characteristic

$$X \mapsto \chi(X) \in \mathbb{Z}.$$

Invariants

Examples

- Ordinary homology and cohomology

$$X \mapsto H_*(X) \qquad \text{or} \qquad H^*(X).$$

  $H_*(X)$ is a graded abelian group and $H^*(X)$ is a graded ring.

- Complex K-theory

$$X \mapsto K^*(X).$$

- Cobordism

Cohomology theories

Topologists' favourite invariants are cohomology theories.
A cohomology theory $E^*(-)$ assigns to each topological space X a $\mathbb{Z}$-graded abelian group $E^*(X)$.
This means that we have an abelian group $E^n(X)$, for each integer n.

- This is done in such a way that $E^*(-)$ shares many of the basic properties of ordinary cohomology.
- In particular, standard techniques for calculation are available.
- Each example provides both a homology theory and a cohomology theory, "dual" to each other.
- Interesting examples have a multiplication: $E^*(X)$ is a graded ring.

Genera

An elliptic cohomology theory will be a cohomology theory somehow associated to an elliptic curve.
To explain how this association takes place, we will need to discuss genera and their associated log series.

A genus is a special sort of invariant.
This is defined for manifolds.

An n-dimensional manifold is a compact space M which is locally homeomorphic to $\mathbb{R}^n$, with some additional structure allowing differentiation.
A manifold with boundary is like a manifold except that some points have neighbourhoods that look like $\mathbb{R}^{n-1} \times [0, \infty)$ rather than $\mathbb{R}^n$.

Genera

Definition

A genus associates to each closed oriented manifold M an element $\Phi(M)$ of a ring R in such a way that:

- $\Phi(M_1 \amalg M_2) = \Phi(M_1) + \Phi(M_2)$, where $\amalg$ denotes disjoint union,
- $\Phi(M_1 \times M_2) = \Phi(M_1)\Phi(M_2)$,
- $\Phi(M_1) = \Phi(M_2)$ if M_1 and M_2 are cobordant.

So now I need to explain what “cobordant” means.

Cobordism

Let M_1, M_2 be closed oriented manifolds of dimension n.

Definition

M_1 and M_2 are cobordant if there is an oriented manifold W of dimension $n+1$ such that $\partial W = M_1 \amalg -M_2$.

This is an equivalence relation and we write MSO_n for the set of equivalence classes.
Then MSO_* (that is the collection of MSO_n for $n \geq 0$) forms a graded ring, where the operations come from disjoint union and Cartesian product:

$$[M_1] + [M_2] = [M_1 \amalg M_2],$$
$$[M_1][M_2] = [M_1 \times M_2].$$

Cobordism

Theorem (Thom)

$MSO_* \otimes \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}][x_1, x_2, x_3, \ldots]$, a polynomial ring on infinitely many variables, with x_i in degree $4i$.

The generators can be taken to be the classes of the even dimensional complex projective spaces $[\mathbb{C}P^{2i}]$.

We can also generalise the above construction, to define a homology theory.

Cobordism as a homology theory

Given a space X, we consider maps $f : M \to X$, imposing the equivalence relation :
$f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are equivalent if there is $F : W \to X$ such that the restriction to the boundary of W gives $f_1 \amalg -f_2 : M_1 \amalg -M_2 \to X$.

Definition

The set of equivalence classes is denoted $MSO_*(X)$.

Then $X \mapsto MSO_*(X)$ is a homology theory.
The first case we considered corresponds to taking X to be a point.

Logs

Now we can reformulate the definition of a genus as a ring map ϕ from MSO_* to a graded ring R_*.
We simplify matters by assuming $\frac{1}{2} \in R_*$ and consider $\phi : MSO_* \otimes \mathbb{Z}[\frac{1}{2}] \to R_*$.
Then, by Thom's theorem, a genus is completely determined by what it does on the generators $x_i = [\mathbb{C}P^{2i}]$.
This is recorded in the associated log series:

$$\log_\phi(x) = \sum_{i \geq 0} \frac{\phi([\mathbb{C}P^{2i}])}{2i+1} x^{2i+1}$$

in $R \otimes \mathbb{Q}[[x]]$.
And, in fact, providing such a log series is equivalent to providing a genus.

From genera to cohomology theories

Given a genus $\phi : MSO_* \to R_*$, this makes R_* into a module over MSO_*.
We can attempt to use this to make a new homology theory, a kind of quotient theory of cobordism, by trying

$$X \mapsto MSO_*(X) \otimes_{MSO_*} R_*.$$

This "algebraic trick" quite often works, and a theorem of Landweber tells us exactly when it does.

The story so far I

$$\begin{array}{ccc} \text{nice cohomology} & \longleftrightarrow & \text{nice genera} \\ \text{theories} & & \text{i.e. ring maps} \\ X \mapsto E^*(X) & & MSO_* \to R_* \end{array}$$

We need to explain how to produce a genus, and so a cohomology theory, from an elliptic curve.

Elliptic curves

Definition

An elliptic curve is the complex plane $\mathbb{C}$ modulo a lattice Λ.

Topologically, this gives a torus.
But $\mathbb{C}/\Lambda$ also has the structure of an abelian group, coming from addition in $\mathbb{C}$.
It also has the structure of a compact Riemann surface - i.e. a compact 1-dimensional complex manifold.
There is a standard way to associate to the lattice a cubic equation of the form

$$y^2 = 4x^3 - g_2 x - g_3,$$

where $g_2 = g_2(\Lambda), g_3 = g_3(\Lambda) \in \mathbb{C}$.

Elliptic cohomology

Let C be an elliptic curve. Write its equation as

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

Let $\delta = -\frac{3}{2}e_1, \epsilon = (e_1 - e_2)(e_1 - e_3)$.
Then, under a standard change of variables, its equation becomes the Jacobi quartic:

$$v^2 = 1 - 2\delta u^2 + \epsilon u^4.$$

Definition

For an elliptic curve C, the associated log series is given by the elliptic integral:

$$\log_{\phi_C}(x) = \int_0^x \frac{1}{(1 - 2\delta t^2 + \epsilon t^4)^{1/2}}\, dt.$$

Elliptic cohomology

The log series corresponds to a genus

$$\phi_C : MSO_* \to \mathbb{Z}[1/2][\delta, \epsilon].$$

If the discriminant $\Delta = \epsilon^2(\delta^2 - \epsilon)$ is non-zero, then this satisfies the conditions necessary to produce a cohomology theory via Landweber's theorem, giving $Ell_C^*(-)$.

The story so far II

nice cohomology theory $\longleftarrow$ nice genus

$Ell_C^*(-)$ $\quad$ $MSO_* \to R_* = \mathbb{Z}[1/2][\delta, \epsilon]$

$\uparrow$

associated log series
defined by elliptic integral

$\uparrow$

elliptic curve
C

Modular forms

The value of elliptic cohomology on a point is closely related to modular forms.

We can think of a lattice Λ in $\mathbb{C}$ as generated by 1 and z for some z in the upper half plane (up to isomorphism of elliptic curves).

If we change z by $z \mapsto \frac{az+b}{cz+d}$, where a, b, c and d are integers and the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 1, we get an isomorphic elliptic curve.

The group of such matrices is $SL(2, \mathbb{Z})$.

Modular forms

Definition

A *modular function of weight n* is an analytic function $f : H \to \mathbb{C}$ on the upper half H of the complex plane such that

$$f((az+b)/(cz+d)) = (cz+d)^n f(z)$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbb{Z})$.

A modular function is a *modular form* if it is well-behaved as $Im(z) \to \infty$.

There are only non-zero modular forms for weights which are natural numbers.

Modular forms

There is a graded ring of modular forms: if you add two modular forms of weight n you get another one of weight n, and if you multiply two modular forms of weights m and n, you get one of weight $m + n$.

There are lots of variants, corresponding to restricting to subgroups of finite index in $SL(2, \mathbb{Z})$.

One variant turns out to give the graded ring $\mathbb{Z}[1/2][\delta, \epsilon]$, with δ in degree 4 and ϵ in degree 8.
It is not a coincidence that this is the same ring that showed up earlier: it's not hard to see that the elliptic genus we defined earlier takes values in this ring of modular forms.

A universal elliptic cohomology?

To an elliptic curve we have associated a cohomology theory.

$$\text{elliptic curve } C \mapsto Ell_C^*(-)$$

Then $Ell_C^*(-)$ is (an) elliptic cohomology.
So there are many elliptic cohomologies, not just one.
We would like to say: take the "universal elliptic curve" and then the associated cohomology theory is the (universal) elliptic cohomology.
Unfortunately, the "universal elliptic curve" doesn't exist.
Nonetheless, it is possible to define a kind of universal elliptic cohomology, called $tmf^*(-)$, short for "topological modular forms".
It's universal in the sense that it comes with a good map to any elliptic cohomology theory.

Its place in a hierarchy

The complicated information available to all cohomology theories can be organised using the chromatic filtration. Here are the bottom levels of the filtration:

Level	Cohomology theory	Geometry
0	$H^*(-;\mathbb{Q})$	points
1	$K^*(-)$	vector bundles
2	Elliptic theories	???

So elliptic theories live at level 2 in this (infinite) hierarchy, representing the next step beyond ordinary cohomology and K-theory.

Conjectural descriptions

Finding a "good" description of elliptic cohomology is a hot topic of current research.

- Answer 1: [Baas-Dundas-Rognes]
  Elliptic cohomology is a "categorification of K-theory"
  Elliptic cohomology should be built out of things called 2-vector bundles, in the same way that K-theory is built out of vector bundles.
- Answer 2: [Segal; Stolz-Teichner]
  Building on ideas of Segal, relating equivariant versions of elliptic cohomology to loop groups, Stolz and Teichner propose that tmf is closely related to supersymmetric conformal field theories.

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