RIGIDITY THEOREMS IN STABLE HOMOTOPY THEORY

CASE FOR SUPPORT

1. Track Records

Andrew Baker was awarded a Ph.D. from the University of Manchester in 1980. He spent 11 years in a succession of postdoctoral positions in Canada, USA and Britain, including two years at the University of Chicago as an L. E. Dickson Instructor and 2 years as an EPSRC Advanced Fellow, before being appointed in 1991 to a Lectureship (and subsequently in 1996 to a Readership) at the University of Glasgow. His research has centred on algebraic topology, especially stable homotopy theory. In particular he has focused on applications of algebra and number theory to complex oriented and periodic cohomology theories (especially K-theory and elliptic cohomology). For a representative overview of his work see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In recent years he has been very involved in work on structured ring spectra and related topics, and organised a series of workshops in Glasgow, Bonn and Rosendal (Norway), as well as editing a book based on the first of these [13].

Sarah Whitehouse was awarded a Ph.D. from the University of Warwick in 1994. She spent several years in France, as a Marie-Curie post-doctoral researcher at the Université Paris-Nord and as a Lecturer at the Université d'Artois. She joined the University of Sheffield as a Lecturer in 2002 and was promoted to Senior Lecturer in 2005. Much of her work has involved the algebras of operations or cooperations of generalised cohomology theories [18, 19, 27, 38]. Recently, this has given new results for complex K-theory, cobordism and the Morava K-theories [15, 16, 21, 17, 35, 36]. She attended all the recent workshops on structured ring spectra mentioned above and she organised a homotopy theory conference in Sheffield in January 2006 at which many of these themes were pursued.

2. Description of Proposed Research

Background

This project is in stable homotopy theory and concerns the "chromatic" approach to this subject. A key feature of the chromatic approach is the study of various *Bousfield localisations* of the homotopy category of spectra. Particularly important are localisations with respect to the *Johnson-Wilson theories* $E(n)$ as well as with respect to the *Morava K-theories* $K(n)$ which play the role of a kind of residue field for $E(n)$. These localised categories can be viewed as approximations to the homotopy category of spectra. Essentially, Bousfield localisation serves to focus attention on the part of stable homotopy theory visible to a given homology theory. These stable homotopy categories have underlying *model categories* and it is an interesting and important question as to whether such structures are *rigid*,

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i.e., whether there are essentially different model structures underlying a given stable homotopy category.

We propose to study questions of rigidity of model structures for various localised categories together with possible connections between rigidity and other phenomena. The main techniques will be those of model categories, including various uniqueness results, together with standard stable homotopy theory methods.

The cohomology theories $K(n)$ were introduced by Morava in the 1970s. That they have come to play a central role in stable homotopy theory is demonstrated, for example, in the work of Hopkins and Smith on maps between finite complexes [22]. For a fixed prime p (usually omitted from the notation), the theories $K(0)$, $K(1)$, $K(2), \ldots$ form a sequence of periodic cohomology theories. $K(0)$ is ordinary rational cohomology and $K(1)$ is one of the $p-1$ isomorphic summands of K-theory with mod p coefficients. The Johnson-Wilson theories $E(n)$ are another important family of cohomology theories. Again there is one for each n and each prime p (with the p omitted from the notation).

For each $n = 0, 1, 2, \ldots$, one may consider the $E(n)$-local category. The localisation functor in this case is denoted L_n. As n increases, the local category becomes more intricate and its homotopy category gives a closer approximation to the full stable homotopy category. This hierarchy of categories provides a powerful tool for studying the structure of stable homotopy theory. Indeed, much of modern stable homotopy theory is related to trying to understand the chromatic tower:

$$\cdots \longrightarrow L_n(X) \longrightarrow L_{n-1}(X) \longrightarrow \cdots \longrightarrow L_1(X) \longrightarrow L_0(X).$$

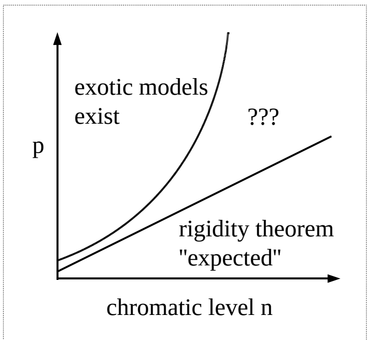
Localisation $\widehat{L}_n$ with respect to $K(n)$ also plays an important role and roughly speaking, $\widehat{L}_n$ measures the difference between L_{n-1} and L_n.

There are many categories of spectra providing point set models for the stable homotopy category, the most useful having the structure of a Quillen closed model category. The main rigidity theorem, due to Schwede and building on earlier work with Shipley, says essentially that all models of the stable homotopy category are Quillen equivalent [30, 31, 32]. This means that they not only have the same homotopy categories, but also that higher order structure such as Toda brackets coincides. More precisely, Schwede's result says that if the homotopy category of a stable model category $\mathcal{C}$ is equivalent as a triangulated category to the homotopy category of spectra, then $\mathcal{C}$ is Quillen equivalent to the model category of spectra. This means that all of the higher order structure, which might be lost on the homotopy level, is already encoded in the triangulated structure of the stable homotopy category.

However, the situation is different for localised categories. The most notable result here is due to Franke [20], building on earlier work of Bousfield [14], which provides a close connection to recent work of one of the investigators [17]. Franke constructs an algebraic abelian model of the $E(n)$-local category when $2p - 2 > n^2 + n$. As an application of a general uniqueness theorem, he proves that its algebraic derived category is equivalent as a triangulated category to the homotopy category of $E(n)$-local spectra. However the underlying model category is not Quillen equivalent to the $E(n)$-local category, see [30]. In this sense, it is an *exotic* model. Franke's work involves categories of diagrams of spectra, indexed over finite partially ordered sets. The condition $2p - 2 > n^2 + n$ ensures sparseness in an Adams spectral sequence. For $n = 1$, Franke's results provide an exotic model for all odd primes p; see [28]. Roitzheim has proved that in the remaining case $p = 2$, the local category is rigid [29].

There are several open questions in this area and the project will address a number of these. Firstly, the question of whether the $E(n)$-local category is rigid remains open whenever $2p-2 \leqslant n^2+n$, except for the one case settled by Roitzheim. In particular, for $n = 2$, the two open cases are $p = 2$ and $p = 3$. Investigation of these cases is likely to involve some new ingredients. In particular, the case when $n = 1$ and $p = 2$ involves use of the Telescope Conjecture which is not known to hold in more general settings, so it would be interesting to understand whether this dependence on the Telescope Conjecture can be eliminated. Similar questions relating to the $K(n)$-local categories will also be investigated.

We aim to gain a more systematic understanding of when rigidity holds or fails to hold. Progress in this direction should be possible by detailed analysis of Schwede's proof of rigidity in the global setting. Particular Toda brackets of elements in the stable homotopy groups of spheres π_*S^0 play a critical role. So rigidity should be related to the behaviour of these Toda brackets in the homotopy of the localised sphere L_nS^0. These considerations lead one to guess that rigidity should hold for p small relative to n, where a linear relation might be expected. This leaves a range, between linear and quadratic in n, where Franke's results do not apply and the global theorem does not shed light. It seems not to be known whether to expect rigidity theorems or exotic models in this range.



Another direction for study is how many exotic models can exist. In the cases where there is an exotic model for the local category, we know of just one such model, namely that constructed by Franke. It would be interesting to understand whether there can be many exotic models and, if so, how one might classify them up to Quillen equivalence. Essentially nothing is known about this question. An answer, even in a very particular case such as $n = 1$, $p = 3$ would be interesting. One can also formulate variants of this question which are perhaps more accessible, such as whether there can be inequivalent *algebraic* models in the form of modules over differential graded algebras. One could try to build a differential graded algebra modelling the homotopy of the local sphere including all Toda brackets. This may allow the introduction of ideas from Morita theory for model categories.

We will also look for connections between rigidity and the non-existence of the Smith-Toda complexes $V(n)$. Smith and Toda [34, 37] considered the existence of finite spectra $V(n)$ with coefficient groups a specified cyclic module $\pi_*(BP)/I_n$ over the coefficients of the Brown-Peterson spectrum $\pi_*(BP)$. The existence or otherwise of these complexes depends on n and on the prime p (which we usually omit from the notation). Various existence and non-existence results were proved by Adams, Smith, Toda and Ravenel [1, 34, 37, 25]. More recently, Nave [24] showed that for each prime $p > 5$, the Smith-Toda complex $V((p+3)/2)$ cannot exist. This improved on an earlier (unpublished) result due to Hopkins, Mahowald and Miller showing that for $p > 3$, $V(p-2)$ does not exist. Thus, roughly speaking, the results so far lead one to expect that $V(n)$ exists if and only if the prime p is sufficiently large compared to n, where sufficiently large means a condition linear in n. There is no known direct connection between rigidity questions and the $V(n)$'s. However, the apparently similar dependence of p on n in the two problems makes it natural to speculate about such a connection. Toda brackets might be a common ingredient. If found, a connection would bring together two different lines of research and would help towards a structural explanation of the two phenomena.

In summary, the global rigidity theorem of Schwede is a striking recent result in stable homotopy. The contrast with the existence of exotic models for certain localised categories leaves many tantalising open questions which this timely project will investigate.

Programme and Methodology

1. The first step is to review the proof of the rigidity theorem in the $n = 1$, $p = 2$ case, carefully identifying which arguments work just for $n = 1$ and which may be easily generalised to higher n. The role of the Telescope Conjecture will be studied with a view to eliminating its use in the proof.

2. A detailed analysis of the proof of Schwede's global rigidity theorem will also be carried out, identifying any implications for rigidity in localised categories. The central part played by Toda brackets will be especially scrutinised with a view to developing analogues in local situations.

3. The next open cases, $n = 2$ for $p = 2$ and $p = 3$ will be investigated. This is likely to involve computations in the homotopy groups of the local sphere L_2S^0. These homotopy groups are known for $p = 3$ [33] but not for $p = 2$.

4. The question of whether there can be many exotic models will be considered, initially in the case $n = 1$, $p = 3$. A first step will be to consider the possibility of multiple algebraic models, possibly with some form of classification.

5. Throughout we will be looking for clues as to the range of rigidity and for a systematic explanation of this.

6. We will look for a link between rigidity and the non-existence of the Smith-Toda complexes $V(n)$, possibly to be formulated in terms of Toda brackets. The work of Nave [23, 24] provides part of the motivation for this.

Dissemination and Beneficiaries

Results of this research will be disseminated through publication in journals devoted to research in pure mathematics, together with presentations at suitable seminars and conferences and electronic distribution of preprints.

This research will interest mathematicians working in algebraic topology, particularly in stable homotopy theory. Aspects of the project may also be of interest to algebraists.

Management

The main cost is the salary of a Research Assistant for 36 months, of which the first 18 months will be spent at the University of Sheffield and the remaining 18 months at the University of Glasgow.

The project is split between Sheffield and Glasgow to allow the RA to benefit from interacting with research groups in both universities and to stimulate collaboration between the two main investigators. The first 18 months in Sheffield will allow the RA to develop mathematically in a very stimulating research environment, since there is a very strong and active group in stable homotopy theory in Sheffield, making it one of the leading European centres for this area of pure mathematics. The move to Glasgow for the second 18 month period brings the opportunity to work with Baker and to benefit from his links with other key European centres for this subject area. Baker already has strong links with Bonn and in 2005 he spent 6 months at the Max-Planck-Institut für Mathematik and the University of Bonn. These links would undoubtedly be further strengthened during the course of the work of this project and we would expect to arrange a number of visits in both directions. Baker also has links to the group of Prof. Rognes in Oslo, where an important programme on the geometry and arithmetic of structured ring spectra will take place in 2007-2009. This is funded by the Norwegian Research Council, in the context of their Outstanding Younger Investigators programme, and Baker will spend five months there in 2007. The RA for this project would also participate in this programme. Altogether, the project will provide an excellent three year training period for the RA.

There are several close connections between the work of the two main investigators. They have previously had many useful research discussions, including during visits to each other's institutions and recently at the Mittag-Leffler Institute in Stockholm. This project will provide a natural focus for joint work.

The Universities of Glasgow and Sheffield will provide normal desk, library and computing facilities.

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