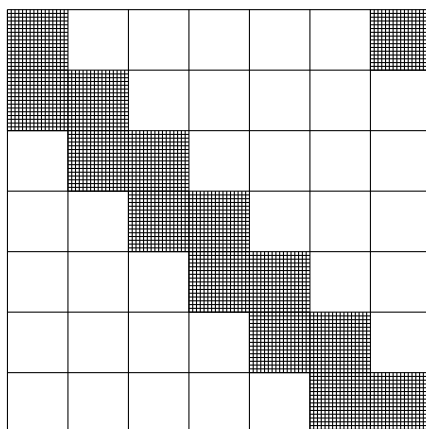


**MAS334 COMBINATORICS 2011/2012**  
**Solutions to Example Sheet 5 : Latin Squares and Designs**

1) Translating to rooks, we need the number of ways of placing  $n$  non-challenging rooks on the  $n \times n$  unshaded board:



Rotating by  $180^\circ$ , this gives the unshaded board of Question Sheet 3, Question 5. So, as in that question, the number is

$$\sum_{k=0}^n (-1)^k (n-k)! \left( \binom{2-k}{k} + \binom{2n-1-k}{k-1} \right).$$

2) Denote by  $L(i)$  the number of  $i$ s appearing in the given  $4 \times 4$  square. We have

$$L(1) = 4, L(2) = 3, L(3) = 2, L(4) = 3, L(5) = 1, L(6) = 2, L(7) = 0.$$

By Theorem 78, a  $p \times q$  Latin rectangle extends to an  $n \times n$  Latin square if and only if  $L(i) \geq p + q - n$  for all  $i$  such that  $1 \leq i \leq n$ .

Taking  $p = q = 4$  and  $n = 7$  we need  $L(i) \geq 4 + 4 - 7 = 1$  for each  $i$  such that  $1 \leq i \leq 7$ . Thus the extension to a  $7 \times 7$  Latin square is possible if and only if  $x = 7$ .

Taking  $p = q = 4$  and  $n = 6$ , we need  $L(i) \geq 4 + 4 - 6 = 2$  for each  $i$  such that  $1 \leq i \leq 6$ . Thus the extension to a  $6 \times 6$  Latin square is possible if and only if  $x = 5$ .

Taking  $x = 5$  and extending by finding distinct representatives of suitable sets, as in the proof of Theorem 78, one possible extension is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 3 & 4 & 6 & 1 & 2 & 5 \\ 4 & 1 & 2 & 5 & 6 & 3 \\ 2 & 3 & 5 & 6 & 4 & 1 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$$

3) The question assumes  $p \leq n$  and  $q \leq n$ . Let  $L(i) = k$  for  $1 \leq i \leq n$ . Clearly,  $\sum_{i=1}^n L(i) = pq$  and so  $kn = pq$ . Thus we have  $k = \frac{pq}{n}$ . By Theorem 78, the extension to an  $n \times n$  Latin square exists if and only if  $k \geq p + q - n$ . Now

$$\begin{aligned} k \geq p + q - n &\iff \frac{pq}{n} \geq p + q - n \\ &\iff pq \geq np + nq - n^2 \\ &\iff n^2 - (p + q)n + pq \geq 0 \\ &\iff (n - p)(n - q) \geq 0. \end{aligned}$$

By assumption  $n \geq p, q$ , so  $k \geq p + q - n$  and by Theorem 78 the extension of  $L$  exists.

4)  $\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array} \quad \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}$

5) We let the blocks consist of all possible ways of choosing  $k$  varieties from the  $v$  varieties. So then we have  $b = \binom{v}{k}$  blocks, each with  $k$  varieties.

Then we need to consider a pair of varieties  $i, j$ . This pair occurs in all blocks where we have  $i, j$  and we can make any choice of the other  $k - 2$  varieties from the  $v - 2$  which are not  $i$  or  $j$ . So each pair occurs in  $\binom{v-2}{k-2}$  blocks. Thus  $\lambda = \binom{v-2}{k-2}$  and we do have a design.

It then follows that each variety is in the same number of blocks  $r$ , with  $r = bk/v = \binom{v}{k}k/v = \binom{v-1}{k-1}$ . (Or, alternatively, each variety appears in  $\binom{v-1}{k-1}$  blocks since there are  $\binom{v-1}{k-1}$  ways to choose the other  $k - 1$  varieties in the block from the other  $v - 1$  varieties.)

6a) Since  $23 = 4.5 + 3$ , by Theorem 90 there is a  $(23, 23, 11, 11, 5)$  design with starter block the quadratic residues of 23. Working mod 23, we have

$$\begin{aligned} 1^2 \equiv 1, & \quad 2^2 \equiv 4, & \quad 3^2 \equiv 9, & \quad 4^2 \equiv 16, & \quad 5^2 \equiv 2, & \quad 6^2 \equiv 13, \\ 7^2 \equiv 3, & \quad 8^2 \equiv 18, & \quad 9^2 \equiv 12, & \quad 10^2 \equiv 8, & \quad 11^2 \equiv 6. \end{aligned}$$

So the quadratic residues are  $\{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$ . Take this as the first block and form each subsequent block by adding 1 and working mod 23.

**b)** The idea is to replace each block by its complement in the set of varieties. Certainly this will give  $b$  new blocks, each block containing  $v - k$  varieties. To check this *is* a design we need to see that each pair appears in the same number of blocks.

We calculate how many new blocks the pair  $(i, j)$  appears in. This happens if and only if neither  $i$  nor  $j$  appeared in the corresponding old block. By the Inclusion/Exclusion Principle (Theorem 31), the number of times this happens is given by

$$\begin{aligned} & \text{no. of old blocks not containing } i & + & \text{no. of old blocks not containing } j & - & \text{no. of old blocks not containing } i \\ & & & & & \text{or not containing } j \\ = & (b - r) & + & (b - r) & - & (b - \lambda) \\ = & b - 2r + \lambda. \end{aligned}$$

Thus we do have a design. Clearly each variety appears in  $b - r$  new blocks, so we have a  $(v, b, b - r, v - k, b - 2r + \lambda)$  design.

**c)** Suppose a  $(v = 23, b = 23, r, k, \lambda)$  design exists. By Theorem 88,  $r = \frac{bk}{v} = \frac{23k}{23} = k$ . Also  $r = \frac{\lambda(v-1)}{k-1}$ , so  $k(k - 1) = 22\lambda$ .

Recall that we are only interested in  $k$  in the range  $1 < k < v = 23$ . Now we need  $k$  such that  $k(k - 1)$  is divisible by  $22 = 2 \cdot 11$ . Below we analyze the only possibilities for  $k$  satisfying these conditions and, in each case, we show that there is a corresponding design. The possibilities are as follows.

1.  $k = 11, k - 1 = 10$ . Then  $\lambda = 5$ . This corresponds to a  $(23, 23, 11, 11, 5)$  design and we constructed such a design in part a).
2.  $k - 1 = 11, k = 12$ . Then  $\lambda = 6$ . This corresponds to a  $(23, 23, 12, 12, 6)$  design. Such a design exists since we can use the complement of the first design, as in part b).
3.  $k = 22, k - 1 = 21$ . Then  $\lambda = 21$ . This corresponds to a  $(23, 23, 22, 22, 21)$  design. Such a design can be constructed as follows. Put all the varieties except  $i$  in block  $i$ . Then we have 23 blocks with 22 varieties per block. Each pair  $(i, j)$  appears in all blocks except blocks  $i$  and  $j$ , so in 21 blocks.

**7a)** In any design, we have  $\frac{bk}{v} = \frac{\lambda(v-1)}{k-1}$ . So for such a design  $b$  must satisfy  $\frac{3b}{11} = \frac{1 \cdot (11-1)}{(3-1)}$ . That is,  $3b = 55$ . But this is impossible since  $b$  must be an integer.

**b)** We have  $r(k - 1) = \lambda(v - 1)$ , with  $k = 3$  and  $\lambda = 1$ . So  $2r = v - 1$  and so  $v$  is odd. So  $v$  is  $1, 3$  or  $5 \pmod{6}$ . On the other hand,  $b(k - 1)k = \lambda(v - 1)v$  gives  $6b = (v - 1)v$ . So 3 divides  $v$  or 3 divides  $v - 1$ . Hence  $v$  must be  $1$  or  $3 \pmod{6}$ .