MAS334 COMBINATORICS 2017/2018

Solutions to Example Sheet 4 : Applications of Hall's marriage theorem

1. The algorithm is indicated below.

$$\begin{array}{ccccc} 0 & 3 & 3 & 4 & 3 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 4 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 & 3 \end{array}$$

0s contained in 3 lines
Lowest not crossed is 1
Take 1 from uncrossed rows
Add 1 to crossed columns

$$\begin{array}{ccccc} 0 & 2 & 3 & 3 & 2 \\ 3 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{array}$$

0s contained in 4 lines
Lowest not crossed is 2
Take 2 from uncrossed rows
Add 2 to crossed columns

$$\begin{array}{ccccc} \boxed{0} & 0 & 1 & 1 & 0 \\ 5 & \boxed{0} & 2 & 0 & 0 \\ 0 & 0 & \boxed{0} & 1 & 0 \\ 2 & 0 & 0 & 1 & \boxed{0} \\ 4 & 0 & 0 & \boxed{0} & 2 \end{array}$$

Jobs with five 0s;
e.g. Aa, Bb, Cc, De, Ed
all with a total of 4 in
the original table.

There are quite a few other choices of optimal allocation - any allocation with a total of 4 in the original table is correct.

2. (i) The sum of the scores is $29 \neq 28 = \binom{8}{2}$, so this is impossible.

(ii) By Landau's Theorem the scores are from a tournament if and only if their sum is $\binom{8}{2} = 28$ and for $1 \leq r \leq 7$ the sum of any r scores is at least $\binom{r}{2}$.

But here, considering the last 4 scores, their sum is 5 and $5 < 6 = \binom{4}{2}$, so they are not the scores of a tournament.

(iii) These are the scores of a tournament, by Landau's Theorem:

The total is $28 = \binom{8}{2}$. The sum of any r is at least the sum of the smallest r, shown in the table and this is at least $\binom{r}{2}$.

r	sum of r smallest scores	$\binom{r}{2}$
1	0	0
2	1	1
3	3	3
4	6	6
5	11	10
6	16	15
7	22	21

Alternatively, part (iii) may be answered by giving a tournament with these scores.

3. (a) Each player plays $n-1$ games, one against each of the other players. Each game results in a win for one of the players. So if player i wins w_i games, he/she loses $n-1-w_i$ games. So $l_i = n-1-w_i$.
(b) Clearly there is a tournament in which the result of every game is the opposite to that in the given tournament. The scores of this tournament are $l_1, \ldots, l_n$.

Alternatively, use Landau's Theorem:

Since $w_1, \ldots, w_n$ are the scores of a tournament, by Landau's theorem, any r of them, say $w_{i_1}, \ldots, w_{i_r}$, add to at least $\binom{r}{2}$. Then

$$\begin{aligned}\sum_{j=1}^{r} l_{i_j} &= \sum_{j=1}^{r} n-1-w_{i_j} = r(n-1) - \sum_{j=1}^{r} w_{i_j} \\ &\leq r(n-1) - \binom{r}{2} \\ &= rn - r - \binom{r}{2} \\ &= rn - \binom{r+1}{2} \\ &= (n-1)+(n-2)+\cdots+(n-r).\end{aligned}$$

Thus any r of the l_is add to at most $(n-1)+(n-2)+\cdots+(n-r)$ and, by Landau's Theorem, these are the scores of a tournament.

4. (a) In a tournament of 3 players, $\binom{3}{2} = 3$ games are played. So there are three scores and the sum of the scores is 3. Each player plays two games, and so each individual score can be $0, 1$ or 2. This gives as potential scores: $2, 1, 0$ and $1, 1, 1$. We can see that both of these really are possible either by just showing tournaments with these scores:

$$\mathbf{A}B, \mathbf{A}C, \mathbf{B}C \qquad \mathbf{A}B, A\mathbf{C}, \mathbf{B}C,$$

or by checking the conditions of Landau's Theorem.
(b) We have to find the number of ways of choosing the other two players subject to the given conditions. These are any two players among the w_i people beaten by player i, so the number is $\binom{w_i}{2}$.
(c) There are $\binom{n}{3}$ ways of choosing a set of three players. By part (a), the scores among these three are either $2, 1, 0$ or $1, 1, 1$. We want to count those with scores $1, 1, 1$, so we subtract the number of those with scores $2, 1, 0$. By part (b), the number of such where player i scores 2 is $\binom{w_i}{2}$. So we get

$$\binom{n}{3} - \binom{w_1}{2} - \binom{w_2}{2} - \cdots - \binom{w_n}{2},$$

as required.

5. (a) The **only** n different non-negative integers less than n are (in decreasing order) $n-1, n-2, \ldots, 2, 1$ and 0. So (uniquely) the scores must be these: a tournament with these scores **is** possible, with (uniquely) P_1 beating all others, P_2 beating all except

P_1, P_3 beating all except P_1 and P_2, etc.
(b) The scores must now be all of $0, 1, 2, \ldots n-1$ except one (i say) and with just one repeated (j say, with $j \neq i$). Their sum $(0+1+2+\ldots+(n-1)) - i + j$ must still equal $\binom{n}{2} = 0+1+2+\ldots+(n-1)$ and so i must equal j, which is a contradiction. So this scenario never happens.
(c) If P_1's score is x and the rest score y (where $x > y$,) then $x + (n-1)y = \binom{n}{2} = \frac{1}{2}n(n-1)$ and so

$$2x = n(n-1) - 2(n-1)y = (n-1)(n-2y).$$

Hence $2x$ is a multiple of $n-1$. But $2x$ is at most $2(n-1)$ and hence $2x = 0$ or $2x = n-1$ or $2x = 2(n-1)$. But $x = 0$ gives $y = \frac{1}{2}n$ (which is bigger than x) and $x = \frac{1}{2}(n-1)$ gives $y = \frac{1}{2}(n-1)$ (which is the same as x). So the only possibility is $x = n-1$ giving $y = \frac{n}{2} - 1$. This means that P_1 wins all his/her games and that n must be even, so that y is an integer.

Finally, we use Landau's theorem to show that the n scores $n-1, \frac{n}{2}-1, \frac{n}{2}-1, \ldots, \frac{n}{2}-1$ **are** possible in a tournament:

We already know that the total of all n scores is correct. If $r \leq n-1$, then any r of the scores add to at least $r\left(\frac{n}{2}-1\right)$. Given that n is even, it's not hard to see that this is at least $\binom{r}{2}$. So the tournament is possible by Landau's Theorem.