MAS334 COMBINATORICS 2017/2018

Solutions to Example Sheet 3 : Rook polynomials

1. Choose the square in the first column of the second row as the specified square s. Then the standard method gives a board C which splits into a product of two boards with polynomials $1+5x+4x^2$ and $1+2x$, and a board D which splits into $1+x$ and $1+4x+2x^2$. Hence, by Theorems 43 and 46, the original board B has rook polynomial

$$(1+2x)(1+5x+4x^2)+x(1+x)(1+4x+2x^2)=1+8x+19x^2+14x^3+2x^4.$$

   Note: Of course, you can choose another square or compute by hand, as long as your method gets the right answer.

2. Either use a direct row-by-row argument or 'split' the board in the following way: label the white squares in odd rows as O and in even rows as E;

O		O	
	E		E
O		O	
	E		E

   Then the O-squares share no row or column with the E-squares. Also both the O-board and the E-board are full $\frac{n}{2}\times\frac{n}{2}$ boards. So the only way to place n non-challenging rooks on the white squares is to have $n/2$ on the O-board and $n/2$ on the E-board. Therefore the n rooks can be placed in $(\frac{n}{2})!^2$ ways.

3. The single square has rook polynomial $1+x$ and so the shaded board (in the $n\times n$ case) has rook polynomial $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n$. A layout of n non-challenging rooks on the unshaded board corresponds to a permutation with $1\to 1$ etc avoided. Hence, using Theorem 51, the number of derangements of $\{1,2,\ldots,n\}$ is

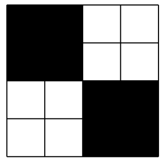
$$\begin{aligned}&n!-(n-1)!\binom{n}{1}+(n-2)!\binom{n}{2}-(n-3)!\binom{n}{3}+\ldots+(-1)^n0!\binom{n}{n}\\=&n!-n!+(n-2)!\frac{n!}{(n-2)!2!}-(n-3)!\frac{n!}{(n-3)!3!}+\ldots+(-1)^n\\=&n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots+(-1)^n\frac{1}{n!}\right).\end{aligned}$$

4. a) This is the rook polynomial of the 1×1 board.

   b) By Theorem 46, this is the rook polynomial of n disjoint 1×1 boards. So, for example, it's the rook polynomial of the shaded part of the board in question 3.

   c) This is the rook polynomial of the full 2×2 board.

   d) So, by Theorem 46, this is the rook polynomial of two disjoint 2×2 boards. So it's the rook polynomial of the unshaded board:



e) This is not a rook polynomial: the coefficients in a rook polynomial are always non-negative integers, since they are the number of ways of placing non-challenging rooks. A coefficient -3 is not possible.

f) This is not a rook polynomial. Since the coefficient of x is 2, there would have to be two unshaded squares on the board. But then the number of ways of placing 2 non-challenging rooks would be 0 or 1; it can't be 2.

5. Placing k non-challenging rooks on the m squares is like seating k people in a row of m seats with no two in adjacent seats. By Example 14, this can be done in $\binom{m+1-k}{k}$ ways: hence this is the coefficient of x^k in the rook polynomial, and the result follows.

6. To find the rook polynomial of the shaded board use the standard method and choose the special square s as the bottom left-hand square. Then B_1 is a staircase of $2n-1$ squares and B_2 is a staircase of $2n-3$ squares. Hence (using the result from question 5) the rook polynomial of the shaded board is

$$\left(\dots + \binom{2n-k}{k}x^k + \dots\right) + x\left(\dots + \binom{2n-1-k}{k-1}x^{k-1} + \binom{2n-2-k}{k}x^k + \dots\right).$$

Hence for the shaded board the coefficient of x^k is $\binom{2n-k}{k} + \binom{2n-1-k}{k-1}$. Therefore, using Theorem 51, the number of ways of placing n rooks on the unshaded board is $\sum_{k=0}^{n}(-1)^k(n-k)!(\binom{2n-k}{k} + \binom{2n-1-k}{k-1})$, and that is the number of ways of seating the wives.