MAS334 COMBINATORICS 2017/2018

Solutions to Example Sheet 2 : Three basic principles

1. (a) In the cases where n is even it is easy to see that such moves are possible; e.g. in each row swap the first piece with the second, the third with the fourth etc.
(b) Picture the board chequered with black and white squares in the usual chess-board fashion. As n is odd there will be more white squares than black, say (with $\frac{1}{2}(n^2+1)$ white and $\frac{1}{2}(n^2-1)$ black). Now each move to an adjacent square is from a black square to a white or *vice-versa.* But there are more white squares than black so not all the pieces moving from the white squares will fit onto the black ones. So such moves are impossible in the odd case.

2. Label pigeon-holes $1, 3, 5, \ldots, 2n-1$ and place each of the $n+1$ numbers into the pigeon-hole corresponding to its highest odd factor. Putting $n+1$ numbers into n pigeon-holes is bound to give two in the same pigeon-hole, by the Pigeon-Hole Principle. So two of the numbers will have the same highest odd factor, f say. So the numbers will be $2^r f$ and $2^s f$ for some integers r and s with $r < s$ say. But then clearly $2^s f = 2^{s-r} \times 2^r f$ and so one is a multiple of the other.

3. Putting each of the m numbers $1, 11, 111, \ldots,$ and $11...1$ (with m digits) into one of the n pigeon-holes corresponding to its remainder when divided by n will give two numbers in the same pigeon-hole, provided $m > n$. That is, two of the numbers have the same remainder when divided by n. So their difference is divisible by n. This difference is clearly of the form $11\ldots10\ldots0$.

4. Label 1000 pigeon-holes with the numbers $0, 1, \ldots, 999$. Put each of the numbers 7^0, 7^1, 7^2, ..., 7^{1000} into the pigeon-hole corresponding to its remainder on division by 1000. There are 1001 integers and 1000 pigeon-holes, so by the Pigeon-Hole Principle there are two numbers in the same pigeon-hole. These two powers of 7 have the same remainder on division by 1000, so they differ by a multiple of 1000.

5. We will use the notation $\lfloor x \rfloor$ to mean the integer part of x. So, for example, $\lfloor 1 \rfloor = \lfloor 1.5 \rfloor = \lfloor 1.999 \rfloor = 1$.
(a) $N(1) =$ number of members of I divisible by $2 = \frac{10000}{2} = 5000$.
$N(1,2) =$ number of members of I divisible by 2 and 3 (i.e. by 6)$=\left\lfloor \frac{10000}{6} \right\rfloor = 1666$.
$N(1,2,\ldots,r) =$ number of members of I divisible by $p_1, \ldots, p_r = \left\lfloor \frac{10000}{p_1 \cdots p_r} \right\rfloor$.
(b) Hence the number of members of I with at least one of the properties 1,2 and 3 is
$$N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3)$$
which is $5000 + 3333 + 2000 - 1666 - 1000 - 666 + 333$ and equals 7334.
(c) Those 7334 numbers include 2, 3 and 5 themselves, but the other 7331 are multiples of these and so not prime. So there are at most 9999 - 7331 = 2668 primes less than ten thousand.

6. There are n^m functions since each of $f(1), f(2), \ldots, f(m)$ has n choices. Now $N(1,2)$ (for example) is the number of functions f from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ for which none of $f(1), f(2), \ldots, f(n)$ equals 1 or 2; i.e. it is the number of functions from $\{1, 2, \ldots, m\}$ to $\{3, 4, \ldots, n\}$, which is $(n-2)^m$. Similarly $N(1,2,3) = (n-3)^m$ etc. Now, by inclusion/exclusion, the number of functions with none of the properties is

$$n^m - N(1) - \cdots - N(n) + N(1,2) + \cdots + N(n-1,n) - \cdots + (-1)^n N(1,2,\ldots,n)$$

which gives the required answer

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \cdots$$

Having none of the properties is equivalent to **all** the numbers in $\{1, 2, \ldots, n\}$ being used by f; i.e. we have counted the **surjections** from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$.

7. There is precisely one solution of the equation for each choice of position for $k-1$ dividers among the $n-1$ positions between n objects (excluding the end positions):

$$\underbrace{\bullet\bullet\bullet}_{x_1}\,|\,\underbrace{\bullet\bullet}_{x_2}\,|\,\bullet\bullet\cdots\bullet\,|\,\underbrace{\bullet\bullet}_{x_k}$$

Therefore there are $\binom{n-1}{k-1}$ solutions.

Let P_1 be the property that $x_1 > 5$, P_2 the property that $x_2 > 10$ and P_3 the property that $x_3 > 15$. We want the number of positive integer solutions with *none* of the properties P_1, P_2, P_3.

So this is

$$\begin{aligned}&\text{total no. of solutions} - \text{no. of solutions with at least one of the properties}\\ &= \binom{19}{2} - \big(N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3)\big),\end{aligned}$$

where we have adopted usual Inclusion/Exclusion notation.

Now, if $x_1 > 5$, write $x_1 = 5 + x_1'$ where x_1' is a positive integer. So positive integer solutions of the original equation with $x_1 > 5$ correspond to positive integer solutions of $x_1' + x_2 + x_3 = 15$. So, by the first part, $N(1) = \binom{14}{2}$.

Similarly, $N(2) = \binom{9}{2}$, $N(3) = \binom{4}{2}$, $N(1,2) = \binom{4}{2}$, and $N(1,3) = N(2,3) = N(1,2,3) = 0$.

So the answer is

$$\binom{19}{2} - \binom{14}{2} - \binom{9}{2} - \binom{4}{2} + \binom{4}{2} = 171 - 91 - 36 = 44.$$