MAS334 COMBINATORICS 2017/2018
Solutions to Example Sheet 1 : The binomial coefficients

1. Recall the Binomial Theorem (Theorem 6):

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n.$$

The first two results required are obtained by simply substituting $x = 1$ and then $x = -1$ into this.

Then adding those first two results gives the sum of the even terms as 2^{n-1}, (since twice the sum of the even terms is 2^n).

Similarly subtracting the two results gives the sum of the odd terms as 2^{n-1} too.

2. The answer is 2^n. You could count the subsets of various sizes: there are $\binom{n}{k}$ subsets of k elements. Then adding all these up for the various k would give (by question 1) 2^n. However it is much quicker to see that a subset is determined by deciding for each of $1, 2, 3, \ldots, n$ whether it is IN the subset or OUT of the subset; i.e. a two-way choice n times, giving 2^n options overall.

3. A rectangle is determined by its sides. These are obtained by choosing any two of the $n + 1$ horizontal lines and any two of the $n + 1$ vertical lines. So the number of rectangles is $\binom{n+1}{2}^2$.

[You are not necessarily expected to spot this two line solution straight away. If not, you should begin by considering small examples. Work out what the answer is in the $n \times n$ case, where $n = 1, 2, 3, 4, 5$, by directly counting. Make sure you do this carefully and accurately, as mistakes here will lead to you failing to spot a pattern. You should get the answers $1, 9, 36, 100, 225$. Now look for a pattern. The first thing you notice is that these numbers are all squares: $1^2, 3^2, 6^2, 10^2, 15^2$. Now you need to see a pattern in the numbers $1, 3, 6, 10, 15$. These you should spot in Pascal's triangle, as $\binom{n+1}{2}$ for $n = 1, 2, 3, 4, 5$. So at this point you *guess* that the general answer is $\binom{n+1}{2}^2$. You still need to find an argument to justify this guess in the general case. But now you know that you need to look in the problem for a choice of two things from $n + 1$ things, twice over. Now you should be able to find the two line argument above.]

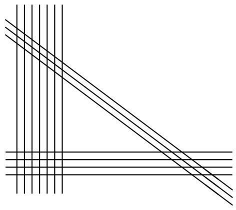
4. (a) Any two lines from the n give rise to a different intersection point, so there are $\binom{n}{2}$ points altogether.
(b) The x_i parallel lines will give no intersection points, resulting in these $\binom{x_i}{2}$ points 'being lost'. So now the number of intersection points is

$$\binom{n}{2} - \binom{x_1}{2} - \ldots - \binom{x_k}{2} = \frac{1}{2}(n^2 - n - x_1^2 + x_1 - \ldots - x_k^2 + x_k) = \frac{1}{2}(n^2 - x_1^2 - \ldots - x_k^2)$$

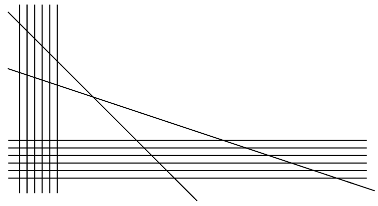
(the last equality being due to the fact that $n = x_1 + \ldots + x_k$).
(c) We need to find some $x_1, x_2, \ldots, x_k$ whose sum is 14 and whose squares sum to 74 (since $\frac{1}{2}(14^2 - x_1^2 - x_2^2 + \ldots - x_k^2) = 61$). Hence the possible values of the x_i's are

$$8, 2, 2, 1, 1 \text{ or } 7, 4, 3 \text{ or } 6, 6, 1, 1;$$

the last two cases are illustrated:



14 lines
61 intersections



5. On the one hand, we may choose k items from n in $\binom{n}{k}$ ways.

   Then, for each of the k items, we may choose to colour the item red or blue, giving 2^k possibilities.

   So there are $2^k\binom{n}{k}$ possibilities in total.

   On the other hand, the number of ways to choose i items to colour red is $\binom{n}{i}$ and the number of ways to choose the remaining $k-i$ items (to be coloured blue) is $\binom{n-i}{k-i}$.

   Therefore the total number of possibilities is $\sum_{i=0}^{k}\binom{n}{i}\binom{n-i}{k-i}$.

   Since these are two ways of counting the same thing, we have the required equality

$$\sum_{i=0}^{k}\binom{n}{i}\binom{n-i}{k-i} = 2^k\binom{n}{k}.$$

6. i)

   a) There are $\binom{2n}{n}$ ways to choose n people from $2n$.
   On the other hand, this may be done by choosing n women from n and 0 men, or $n-1$ women and 1 man, or $n-2$ women and 2 men, and so on.
   The number of ways to do this is

$$\binom{n}{n}\binom{n}{0}+\binom{n}{n-1}\binom{n}{1}+\binom{n}{n-2}\binom{n}{2}+\cdots+\binom{n}{0}\binom{n}{n}.$$

   Using $\binom{n}{i}=\binom{n}{n-i}$, this gives

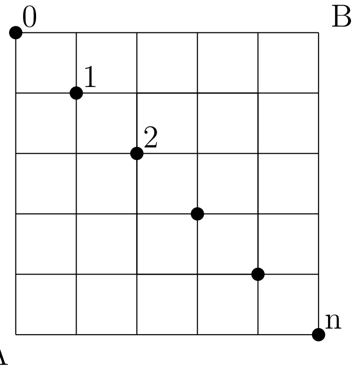
$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}.$$

   b) $\binom{2n}{n}$ is the coefficient of x^n in the expansion of $(1+x)^{2n}$.

$$\begin{aligned}(1+x)^{2n} &= ((1+x)^n)^2\\ &= \left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)^2.\end{aligned}$$

   So the coefficient of x^n is $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$.
   Using $\binom{n}{i}=\binom{n}{n-i}$, this gives $\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$.

   c) The number of shortest possible routes from bottom left point A to top right point B in a square $n\times n$ grid is $\binom{2n}{n}$.
   Each such route passes through precisely one of the $n+1$ points, labelled $0,1,2,\ldots,n$ say, on the diagonal of the grid from top left to bottom right.



The no. of routes through point i

$$= (\text{no. of routes from } A \text{ to } i) \cdot (\text{no. of routes from } i \text{ to B})\,.$$

Routes from A to i consist of $n-i$ units up and i units right and so the number of these is $\binom{n}{i}$.

Similarly, to get from i to B requires going i units up and $n-i$ units right, so again there are $\binom{n}{i}$ such routes.

So the total number of routes is $\sum_{i=0}^{n}\binom{n}{i}^2$.

ii) We may choose i women and i men, for $0 \le i \le n$.

There are $\binom{n}{i}\binom{n}{i}$ ways to choose i women and i men.

So the total is $\sum_{i=0}^{n}\binom{n}{i}^2$.

By the previous parts, this is $\binom{2n}{n}$.

iii) There must be at least one person in the subset to act as leader. For $1 \le i \le n$, there are $\binom{n}{i}$ ways to choose an i person subset and then i possibilities for leader of the subset. So the total is $\sum_{i=1}^{n} i\binom{n}{i}$.

On the other hand, we may first pick a leader, in n ways, and then pick any subset of the remaining $n-1$ people for them to lead, in 2^{n-1} ways.

[Alternative answer for iii):

$$\begin{aligned}
\sum_{i=1}^{n} i\binom{n}{i} &= \sum_{i=1}^{n} i\frac{n!}{i!(n-i)!} \\
&= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \\
&= n\sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!((n-1)-(i-1))!} \\
&= n\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \\
&= n(1+1)^{n-1} = n2^{n-1}.\ ]
\end{aligned}$$