MAS334 COMBINATORICS 2011/2012

Example Sheet 5 : Latin squares and designs

1. In how many ways can the Latin rectangle

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & 1 & 2 & \dots & n-1 \end{pmatrix}$$

be extended to a $3 \times n$ Latin rectangle with entries from $\{1, 2, \dots, n\}$?
(This should *not* require much new effort - use the techniques of Sheet 3.)

2. For what value of x can the following Latin rectangle be extended to a 7×7 Latin square?
For what value of x can it be extended to a 6×6 Latin square?
Write down one such extension to a 6×6 Latin square.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 1 & 2 \\ 3 & 4 & 6 & 1 \\ 4 & 1 & 2 & x \end{pmatrix}$$

3. Let p, q and n be positive integers with $p \leq n$ and $q \leq n$. Let L be a $p \times q$ Latin rectangle in which each of the numbers $\{1, 2, \dots, n\}$ occurs the same number of times. Show that L can be extended to an $n \times n$ Latin square.

4. Write down two orthogonal 3×3 Latin squares.

5. Given integers v and k with $1 < k < v$ show that there exists a $\left(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2}\right)$ design.

6. (a) Explain briefly how to construct a $(23, 23, 11, 11, 5)$ design.
(b) Show that if a (v, b, r, k, λ) design exists (and $k \neq v - 1$) then there also exists a $(v, b, b - r, v - k, b - 2r + \lambda)$ design. [Hint: the idea is to replace each block by its complement in the set of varieties. You need to check that this does result in a design.]
(c) Show that, if a $(23, 23, r, k, \lambda)$ design exists, then $r = k$ and $k(k - 1) = 22\lambda$. Hence find all values of r, k and λ such that a $(23, 23, r, k, \lambda)$ design exists. (Remember that to be sure that one does exist you must explain briefly how to construct it.)

7. (a) Show that there cannot be a design with $k = 3$, $\lambda = 1$ and $v = 11$.
(b) Show that if a design has $k = 3$ and $\lambda = 1$, then v must be congruent to 1 or 3 mod 6.