PMA313 COMBINATORICS — AUTUMN SEMESTER 2009-2010
EXAM SOLUTIONS AND MARK SCHEME

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Solution to Question 1

All parts are unseen apart from part (iii)(a) which is a standard problem.

(i)(a) **(Unseen)**

$$\begin{aligned}\binom{r+2}{3}-\binom{r}{3} &= \frac{(r+2)(r+1)r}{3.2.1}-\frac{r(r-1)(r-2)}{3.2.1} && \textbf{(1 Mark)}\\ &= \frac{r\left((r+2)(r+1)-(r-1)(r-2)\right)}{6} && \textbf{(1 Mark)}\\ &= \frac{r(r^2+3r+2-r^2+3r-2)}{6}=\frac{6r^2}{6}=r^2. && \textbf{(1 Mark)}\end{aligned}$$

(b) **(Unseen)**

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{n}\left(\binom{i+2}{3}-\binom{i}{3}\right) && \textbf{(1 Mark)}\\ &= \sum_{i=3}^{n+2}\binom{i}{3}-\sum_{i=1}^{n}\binom{i}{3} && \textbf{(1 Mark)}\\ &= \binom{n+1}{3}+\binom{n+2}{3}+\sum_{i=3}^{n}\binom{i}{3}-\sum_{i=3}^{n}\binom{i}{3}\\ &= \binom{n+1}{3}+\binom{n+2}{3} && \textbf{(1 Mark)}\\ &= \frac{(n+1)n(n-1)}{6}+\frac{(n+2)(n+1)n}{6} && \textbf{(1 Mark)}\\ &= \frac{(n+1)n(n-1+n+2)}{6}\\ &= \frac{n(n+1)(2n+1)}{6}. && \textbf{(1 Mark)}\end{aligned}$$

(ii) **(Unseen)** Suppose that $n(n+1)|200k$. Then, since n and $n+1$ are coprime to k we find that $n(n+1)|200$ **(1 Mark)** . Note that $200 = 2^3 \times 5^2$ **(1 Mark)** and that one of n, $n+1$ is odd **(1 Mark)** . It follows that either $n|5^2$ or $n+1|5^2$ and hence, by consideration of the fact that $n(n+1) < 200$, either $n = 5$ or $n+1 = 5$ **(1 Mark)** . In the first case, $n(n+1) = 30 \nmid 200$ so we must have $n = 4$ **(1 Mark)** (giving $n(n+1) = 20$).

[**Alternative:** Suppose that $n(n+1)|200k$. Then, since n and $n+1$ are coprime to k we find that $n(n+1)|200$, so $n|200$ and $n+1|200$. Hence we are looking for two consecutive factors of 200. The list of factors of 200 is 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200 and, since $n > 1$, the only option is $n = 4$ and $n+1 = 5$.]

[**Alternative:** Suppose that $n(n+1)|200k$. Then, since n and $n+1$ are coprime to k we find that $n(n+1)|200$ **(1 Mark)** . Thus $n < \sqrt{200}$ so we have $1 < n \leq 14$ **(1 Mark)** . By calculation we find that in this range, the only value of n for which $n(n+1)$ divides 200 is $n = 4$ **(1 Mark)** .

n	$n(n+1)$	factor of 200?
2	6	N
3	12	N
4	20	Y
5	30	N
6	42	N
7	56	N
8	72	N
9	90	N
10	110	N
11	132	N
12	156	N
13	182	N
14	210	N

(2 Marks for table or similar valid reasoning)]

(iii)(a) (**Standard problem**) Any such rectangle is determined by a choice of two vertical sides and two horizontal sides **(1 Mark)** (and each such choice uniquely determines a rectangle). There are $\binom{n+1}{2}$ ways to pick the horizontal sides and $\binom{n+1}{2}$ ways to choose the vertical sides **(1 Mark)** . Thus there are $\binom{n+1}{2}^2$ **(1 Mark)** $= \left(\frac{(n+1)n}{2}\right)^2 = \frac{1}{4}n^2(n+1)^2$ **(1 Mark)** such rectangles.

(b) (**Unseen**) Let $1 \leq k \leq n$ and consider squares in the grid of size $k \times k$. There are $n+1-k$ choices of vertical grid lines for the left of the square and $n+1-k$ choices of horizontal grid lines for the top of the square **(1 Mark)** . Hence there are $(n+1-k)^2$ squares of size $k \times k$ **(1 Mark)** . Thus, the total number of squares in the grid is

$$\sum_{k=1}^{n}(n+1-k)^2 \ \textbf{(1 Mark)} \ = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \ \textbf{(1 Mark)} \ ,$$

using (i)(b).

(c) (**Unseen**) Using parts (a) and (b), the percentage of rectangles that are square is given by

$$\frac{\frac{1}{6}n(n+1)(2n+1)}{\frac{1}{4}n^2(n+1)^2} \times 100 = \frac{200(2n+1)}{3n(n+1)} \ \textbf{(1 Mark)} \ .$$

Note that $2n+1$ is coprime to both n and $n+1$ **(1 Mark)** . If the percentage is an integer, we must have $n(n+1)|200(2n+1)$ and so, by part (ii), $n = 4$ **(1 Mark)** . When $n = 4$ we get a proportion of precisely 30% **(1 Mark)** .

Solution to Question 2

(i)(a) **(Bookwork)** Suppose we have a finite set of items and properties $1, 2, 3, \ldots, n$. Let $N(i_1, i_2, \ldots, i_r)$ be the number of items which have the properties $i_1, i_2, \ldots, i_r$ (and maybe others). Then the number of items with at least one of the properties is

$$\begin{aligned}
&N(1) + N(2) + N(3) + \cdots + N(n) \\
&- N(1,2) - N(1,3) - \cdots - N(n-1,n) \\
&+ N(1,2,3) + N(1,2,4) + \cdots \\
&- N(1,2,3,4) - \cdots \\
&\qquad\vdots \\
&+ (-1)^{n-1} N(1,2,3,\ldots,n).
\end{aligned}$$

(3 Marks)

(b) **(Bookwork)** Such solutions are in one-to-one correspondence with the "shortest routes" from bottom left to top right in a $k \times 2$ rectangular grid: we proceed x units along the bottom line of the grid, up one unit, y units along the middle horizontal line of the grid, one unit up and the remaining z units along the top line.

(2 Marks)

The number of such routes is the number of ways of choosing the positions of 2 steps up from $k+2$ steps altogether, $\binom{k+2}{2}$. **(1 Mark)**

(c) **(Standard problem)** Let Property 1 be that $x \geq 4$, Property 2 be that $y \geq 7$ and Property 3 be that $z \geq 12$. We want the number of positive integer solutions with *none* of the properties 1, 2 and 3. **(2 Marks)**
So this is

total no. of solutions – no. of solutions with at least one of the properties

$$= \binom{20}{2} - (N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3)),$$

where we have adopted usual I/E notation. **(2 Marks)**
Now, if $x \geq 4$, write $x = 4 + x'$ where x' is a non-negative integer. So non-negative integer solutions of the original equation with $x \geq 4$ correspond to non-negative integer solutions of $x' + y + z = 14$. So, by part (i), $N(1) = \binom{16}{2}$.
(2 Marks)
Similarly, $N(2) = \binom{13}{2}$, $N(3) = \binom{8}{2}$, $N(1,2) = \binom{9}{2}$, and $N(1,3) = \binom{4}{2}$, $N(2,3) = N(1,2,3) = 0$. **(2 Marks)**

So the answer is

$$\binom{20}{2} - \binom{16}{2} - \binom{13}{2} - \binom{8}{2} + \binom{9}{2} + \binom{4}{2} = 190 - 120 - 78 - 28 + 36 + 6 = 6.$$

(1 Mark)

(ii)(a) **(Bookwork)** If you place more than n letters in n pigeon-holes then some pigeon-hole will contain more than one letter. **(2 Marks)**

(b) **(Standard problem)** Label 1000 pigeon-holes with the numbers $0, 1, \ldots, 999$. **(1 Mark)**

Put each of the numbers n^0, n^1, n^2, ..., n^{1000} into the pigeon-hole corresponding to its remainder on division by 1000. **(1 Mark)**

There are 1001 integers and 1000 pigeon-holes, so by the Pigeon-Hole Principle there are two numbers in the same pigeon-hole. **(1 Mark)**

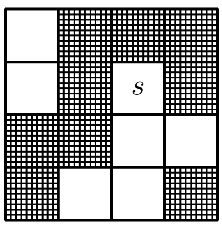
These two powers of n have the same remainder on division by 1000, so they differ by a multiple of 1000. **(1 Mark)**

(c) **(Unseen)** Apply part b) with $n = 13$ to get non-negative integers $s < t$ such that $13^t - 13^s = 13^s(13^{t-s} - 1)$ is divisible by 1000. **(2 Marks)**

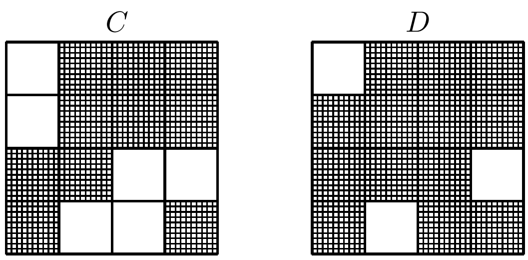
Since 13^s and 1000 are coprime, $13^{t-s} - 1$ is divisible by 1000 and so 13^{t-s} ends in the digits 001. **(2 Marks)**

Solution to Question 3

(i) (**Standard problem**) Here's one method using the relevant theorems on rook polynomials from the course. Label the board B. Begin with the square marked s below and apply Theorem 42.



This gives $r_B(x) = r_C(x) + x r_D(x)$, where C and D are as below.



(2 Marks)

Applying Theorem 45 with some direct counting gives

$$r_C(x) = (1 + 2x)(1 + 4x + 3x^2) \quad \text{and} \quad r_D(x) = (1 + x)^3.$$

(3 Marks)

Hence,

$$\begin{aligned} r_B(x) &= (1+2x)(1+4x+3x^2)+x(1+x)^3 \\ &= (1+6x+11x^2+6x^3)+(x+3x^2+3x^3+x^4) \\ &= 1+7x+14x^2+9x^3+x^4. \end{aligned}$$

(2 Marks)

[**Alternative:** Direct counting by hand will be given full marks if the correct answer is obtained; partial marks may be obtained for a partially correct answer depending on the explanation given for the counting procedure adopted.]

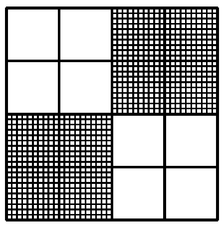
Using Theorem 50 from the notes, we find that the number of ways of placing 4 non-challenging rooks on the shaded board is

$$4! \times 1 - 3! \times 7 + 2! \times 14 - 1! \times 9 + 1 = 2. \qquad (\textbf{2 Marks})$$

(ii)(a) (**Unseen**) First, choose the k rows from the n available in $\binom{n}{k}$ ways **(1 Mark)** . Next, there are n places that a rook can be put in the first chosen row, then $n-1$ places left in the second row, and so on, making $\frac{n!}{(n-k)!} = \frac{n!}{k!(n-k)!}k! = \binom{n}{k}k!$ ways to allocate the positions **(1 Mark)** . Hence, in total, we get $\binom{n}{k}^2 k!$ ways to place the k rooks **(1 Mark)** . Thus

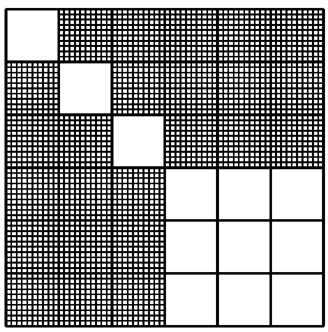
$$r_{F_n}(x) = \sum_{k=0}^{n} \binom{n}{k}^2 k! x^k. \ (\textbf{1 Mark})$$

(b) (**Standard problem**) The obvious choice for B is a $2n \times 2n$ board with 2×2 unshaded blocks down the diagonal, as illustrated for $n = 2$ below.



(2 marks)

The obvious choice here is illustrated below.



(2 Marks)

(c) (**Unseen**) On F_n, label the square corresponding to the one missing from D as s and apply Theorem 42 to get

$$r_{F_n}(x) = r_D(x) + x r_E(x), \text{ } (\textbf{1 Mark})$$

where E is a board with one row and one column fully deleted. It is clear that E has the same polynomial as the full $(n-1)\times(n-1)$ board, that is $r_E(x) = r_{F_{n-1}}(x)$ **(1 Mark)** and, rearranging,

$$r_D(x) = r_{F_n}(x) - x r_{F_{n-1}}(x). \text{ } (\textbf{1 Mark})$$

Thus we have

$$\begin{aligned}
r_D(x) &= \sum_{k=0}^{n} \binom{n}{k}^2 k! x^k - x \sum_{k=0}^{n-1} \binom{n-1}{k}^2 k! x^k \text{ } (\textbf{1 Mark}) \\
&= \sum_{k=0}^{n} \binom{n}{k}^2 k! x^k - \sum_{k=0}^{n-1} \binom{n-1}{k}^2 k! x^{k+1} \\
&= \sum_{k=0}^{n} \binom{n}{k}^2 k! x^k - \sum_{k=1}^{n} \binom{n-1}{k-1}^2 (k-1)! x^k \text{ } (\textbf{1 Mark}) \\
&= \sum_{k=0}^{n} \left(\binom{n}{k}^2 k! - \binom{n-1}{k-1}^2 (k-1)! \right) x^k \text{ } (\textbf{1 Mark})
\end{aligned}$$

(noting that $\binom{n-1}{k-1}^2 (k-1)! = 0$ when $k=0$). Applying Theorem 50 from the notes we see that the number of ways of placing n rooks on the complement of D is

$$\sum_{k=0}^{n} (-1)^k (n-k)! \left(\binom{n}{k}^2 k! - \binom{n-1}{k-1}^2 (k-1)! \right). \text{ } (\textbf{1 Mark})$$

Since the complement of D only has one available square and $n > 1$, we see that this quantity is equal to zero. **(1 Mark)**

Solution to Question 4

(i) **(Bookwork)** The Latin rectangle is extendable iff $L(i) \geq p+q-n$ for $1 \leq i \leq n$. **(1 Mark)**
Assume the rectangle L is extendable. Consider x_i, the number of times i occurs in the first p rows of the extension but not in L itself. **(1 Mark)**
Since i occurs exactly once in every row, it occurs p times altogether in the first p rows. But it occurs $L(i)$ times in L. So $x_i = p - L(i)$. **(1 Mark)**
But this block has $n-q$ columns and since each i needs its own column, we must have $p - L(i) \leq n - q$, i.e. $L(i) \geq p + q - n$. **(2 Marks)**

ii) **(Routine problem)** Here $p = q = 4$ and $n = 6$, so we need $L(i) \geq 2$ for $1 \leq i \leq 6$. This happens iff $x = 4$. **(2 Marks)**

One extension is

$$\begin{pmatrix} 1 & 6 & 5 & 3 & 2 & 4 \\ 2 & 3 & 6 & 5 & 4 & 1 \\ 5 & 4 & 3 & 6 & 1 & 2 \\ 4 & 5 & 2 & 1 & 3 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \\ 6 & 2 & 1 & 4 & 5 & 3 \end{pmatrix} \qquad \textbf{(4 Marks)}$$

[Marking: 2 marks for correct extension to 4×6 or 6×4; 2 marks for extension from there to 6×6.]

(iii) **(Unseen problem)**
The number of times 1 (for example) appears is n. **(1 Mark)**
It appears an even number of times (symmetrically) off the diagonal and once on the diagonal. **(1 Mark)**
So it appears an odd number of times altogether and n is odd. **(1 Mark)**

(iv)(a) **(Bookwork)** Let x be one of the varieties and assume x is in r_x blocks. **(1 Mark)**
Consider all the blocks and count the number of *pairs* that involve x. **(1 Mark)**
Firstly, since x can be paired with $k-1$ others in each of its blocks, this total is $r_x(k-1)$. **(1 Mark)**
On the other hand, there are $v-1$ varieties apart from x. Each pair of x with one other appears λ times in the design. So the total is $\lambda(v-1)$. **(1 Mark)**
So $r_x(k-1) = \lambda(v-1)$ and thus $r_x = \frac{\lambda(v-1)}{k-1}$. **(1 Mark)**
This is independent of x and we call it r. Now, since each variety occurs in r blocks, the total number of entries in all blocks is vr. **(1 Mark)**
On the other hand, we have b blocks of k entries, so this is also bk. So $vr = bk$ and thus $r = \frac{bk}{v}$. **(1 Mark)**

(b) (**Bookwork**) Working mod p, calculate

$$1^2, \quad 2^2, \ldots, (2n+1)^2. \text{ } \textbf{(1 Mark)}$$

By a result in the course, there is a $(4n+3, 4n+3, 2n+1, 2n+1, n)$ design with starter block given by the $2n+1$ quadratic residues calculated above and other blocks obtained by adding 1 to each entry and working mod p. **(2 Marks)**
When $p = 11$ we calculate the quadratic residues to get a suitable starter block of

$$\{1,\ 4,\ 9,\ 5,\ 3\}.$$

(1 Mark)

Solution to Question 5

[(i)(a) **(Bookwork)** A set of women can always find husbands from amongst the men they know if and only if for each r, any set of r of the women know at least r men between them.**2 marks**

(b) Consider the sets $A_1, \dots, A_n$ as women, and their members as the men they know. **(1 Mark)** View a marriage as a choice of element from a set **(1 Mark)** . Then a rephrasing of Hall's Theorem tells us that the sets have distinct representatives if and only if for each k, any choice of k of the sets contain at least k elements between them **(1 Mark)** ; that is, if and only if for any choice of subset $I \subseteq \{1, \dots, n\}$ the union $\cup_{i \in I} A_i$ contains at least $|I|$ elements. **(1 Mark)**

(c) Suppose that $x_1, \dots, x_4$ are all different. Then we can choose x_1 from A_1, x_4 from A_2, x_2 from A_3 and x_3 from A_4 and we have distinct representatives. **(1 mark)**

On the other hand, suppose that the sets $A_1, \dots, A_4$ have distinct representatives. Then, using part (b),

$$|\{x_1, x_2, x_3, x_4\}| = |A_1 \cup A_2 \cup A_3 \cup A_4| \geq 4.$$

Hence x_1, x_2, x_3 and x_4 must all be different. **(2 marks)**

(ii) **(Standard problem) (standard problem)** 2 lines will cover the 0's:

2	0	0	4
3	4	0	2
1	0	1	5
5	2	0	6

(1 Mark)

We identify the lowest number not crossed out, 1. **(1 Mark)**

Then we subtract 1 from every uncrossed row and add 1 to every crossed column. **(1 Mark)**

This gives:

1	0	0	3
2	4	0	1
0	0	1	4
4	2	0	5

(1 Mark)

Now repeat the procedure, to give

0	0	0	2
1	4	0	0
0	1	2	4
3	2	0	4

(1 Mark)

This has 4 0's with no two on the same line and has best allocation in the same position as the original table. **(1 Mark)**
So for example, best allocation is given by Ab, Bd, Ca, Dc, with total unsuitability, taken from the original table, of 3. **(1 Mark)**

(iii)(a) Label the students' scores $s_1, \ldots, s_k$ and the parents' scores $p_1, \ldots, p_k$ for some positive integer k. Then, since each parent wins at most one game, $k \geq p_1 + \ldots + p_k$ **(1 Mark)** . But, by Landau's theorem, $p_1 + \ldots + p_k \geq \binom{k}{2}$ **(1 Mark)** . Hence we have $k \geq \frac{1}{2}k(k-1)$ which rearranges to give $k(k-3) \leq 0$ so that $0 \leq k \leq 3$. **(1 Mark)**

(b) Consider the lowest three scores. Since these must total at least 3 (Landau's Theorem) and each parent's score is at most one, it follows that the lowest three scores must be 1, 1 and 1 (the other option, $0, 1, 2$, isn't possible).**(2 marks)**

Another application of Landau's Theorem shows the next score must be at least 3. **(1 Mark)** Also, Landau's theorem says the top three scores must total no more than 12, meaning the third highest score must be no more than 4. Hence the possibilities are

$$1,\ 1,\ 1,\ 3,\ 4,\ 5 \text{ and } 1,\ 1,\ 1,\ 4,\ 4,\ 4.$$ **(2 marks)**

Finally, since the champion won more games than anyone else, we see that we must have the former of the two above sets; that is, the scores were 1, 1, 1, 3, 4 and 5. **(1 Mark)**