The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2009–10**

Combinatorics **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

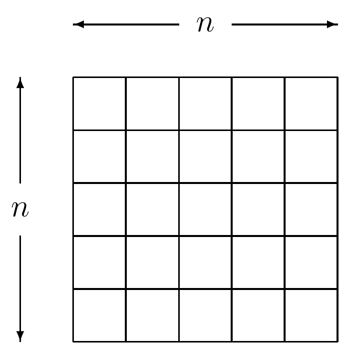
1 (i) (a) Show that $\binom{r+2}{3} - \binom{r}{3} = r^2$ for all non-negative integers r.

(3 marks)

(b) Deduce that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$. ***(5 marks)***

(ii) Let $n > 1$ be an integer and let k be an integer sharing no common factors with n or $n+1$. Show that if $n(n+1)$ divides $200k$ then $n = 4$. ***(5 marks)***

(iii) Consider a grid of n rows and n columns (as illustrated in the case $n = 5$):



(a) How many rectangles can be found which use all or part of some of those grid lines as their boundaries? Justify your answer, which should be simplified as far as possible. ***(4 marks)***

(b) By considering the various sizes (or otherwise) calculate how many of the rectangles found in part (a) are squares. ***(4 marks)***

(c) For one particular grid, with $n > 1$, the percentage of the rectangles which are squares is equal to an integer. Find the value of n and the corresponding percentage, showing that no others work.

(4 marks)

2 (i) (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

(b) Let k be a positive integer. Explain briefly why there are $\binom{k+2}{2}$ non-negative integer solutions of the equation

$$x + y + z = k.$$

(3 marks)

(c) Use the Inclusion/Exclusion Principle to find the number of non-negative integer solutions of the equation

$$x + y + z = 18,$$

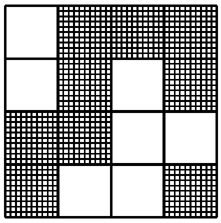
satisfying the conditions $x < 4$, $y < 7$ and $z < 12$. ***(9 marks)***

(ii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Let n be any positive integer. Show that there exist two different powers of n whose difference is divisible by 1000. ***(4 marks)***

(c) Deduce that there is a power of 13 ending with the digits 001. ***(4 marks)***

3 (i) Calculate the rook polynomial of the unshaded board below.



Hence, or otherwise, find the number of ways of placing 4 non-challenging rooks on the **shaded** board above. ***(9 marks)***

(ii) (a) For each positive integer n, let F_n be the full $n \times n$ board. By choosing the rows first and then the positions in the rows, show that k non-challenging rooks can be placed on F_n in $\binom{n}{k}^2 k!$ ways. Hence write down the rook polynomial of F_n. ***(4 marks)***

(b) Draw a board B with rook polynomial $(1 + 4x + 2x^2)^n$ and a board C with rook polynomial $(1 + x)^3(1 + 9x + 18x^2 + 6x^3)$. ***(4 marks)***

(c) Now let $n > 1$ and let D be the full $n \times n$ board with just one square removed. By applying the standard method to F_n, show that

$$r_D(x) = r_{F_n}(x) - x r_{F_{n-1}}(x).$$

Hence find the rook polynomial of D. Deduce that

$$\sum_{k=0}^{n} (-1)^k (n-k)! \left(\binom{n}{k}^2 k! - \binom{n-1}{k-1}^2 (k-1)! \right) = 0.$$

(8 marks)

4 (i) Let L be a $p \times q$ Latin rectangle with entries in $\{1, 2, \ldots, n\}$. For $1 \leqslant i \leqslant n$ let $L(i)$ be the number of occurrences of i in L. State conditions on the $L(i)$ which are necessary and sufficient for L to be extendable to an $n \times n$ Latin square. Prove that the conditions are necessary. ***(5 marks)***

(ii) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 6 & 5 & 3 \\ 2 & 3 & 6 & 5 \\ 5 & x & 3 & 6 \\ 4 & 5 & 2 & 1 \end{pmatrix}$$

Write down one such extension. ***(6 marks)***

(iii) Let $A = (a_{ij})$ be an $n \times n$ Latin square. Suppose that $A = A^T$ and that every element of $\{1, 2, \ldots, n\}$ appears as a diagonal entry a_{ii} of A. Show that n is odd. ***(3 marks)***

(iv) (a) Given a design of v varieties, b blocks, k varieties per block and every pair of varieties occurring in λ blocks, prove that each variety is in r blocks, where

$$r = \frac{bk}{v} = \frac{\lambda(v-1)}{k-1}.$$

(7 marks)

(b) Let p be a prime of the form $4n + 3$. Explain how to construct a $(4n+3, 4n+3, 2n+1, 2n+1, n)$ design. Give a suitable starter-block for such a design when $p = 11$. ***(4 marks)***

5 (i) (a) State (without proof) the marriage version of Hall's Theorem.

(2 marks)

(b) Use the marriage version of Hall's Theorem to deduce that sets $A_1, \ldots, A_n$ have distinct representatives if and only if

$$\left|\bigcup_{i \in I} A_i\right| \geqslant |I| \quad \text{for each } I \subseteq \{1, \ldots, n\}.$$

(4 marks)

(c) Let $x_1,\ x_2,\ x_3$ and x_4 be positive integers. Show that the sets A_1, A_2, A_3 and A_4 below have distinct representatives if and only if $x_1,\ x_2,\ x_3$ and x_4 are all different.

$$\begin{aligned} A_1 &= \{x_1, x_2, x_3\} \\ A_2 &= \{x_2, x_4\} \\ A_3 &= \{x_2, x_3\} \\ A_4 &= \{x_2, x_3\} \end{aligned}$$

(3 marks)

(ii) Jobs a, b, c, d are to be allocated to people A, B, C, D with each person getting one of the jobs. Their unsuitability for the jobs is shown in the following table, lower numbers meaning that people are more suitable for jobs. Use the Hungarian algorithm to allocate the jobs with lowest total unsuitability.

	a	b	c	d
A	2	0	0	4
B	3	4	0	2
C	1	0	1	5
D	5	2	0	6

(7 marks)

(iii) A school tournament took place. An equal number of students and parents entered. Every person played each of the others once. No parent won more than one game and the champion won more games than anyone else.

(a) Show that there were at most six people in the tournament.

(3 marks)

(b) Given that there were exactly six people in the tournament, list the scores.

(6 marks)

End of Question Paper