MAS334 COMBINATORICS — AUTUMN SEMESTER 2010-2011
EXAM SOLUTIONS AND MARK SCHEME

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Solution to Question 1

ia) **(bookwork)** For $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \qquad (\textbf{2 Marks})$$

b) **(unseen)** Differentiating both sides of the binomial identity gives

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}. \qquad (\textbf{2 Marks})$$

Setting $x = 1$ then gives the required identity. **(2 Marks)**

iia) **(unseen)** Three uses of Pascal's Identity give:

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k}\\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k}\\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \binom{n-3}{k}. \qquad (\textbf{3 Marks})\end{aligned}$$

And rearranging gives the result as stated. **(1 Mark)**

b) **(unseen)** As indicated, we consider the number, N say, of subsets of S of size k containing at least one of a, b or c.

On the one hand, there are $\binom{n}{k}$ subsets of S of size k. **(1 Mark)**

And, of these, $\binom{n-3}{k}$ do not contain a, b or c. **(1 Mark)**

So $N = \binom{n}{k} - \binom{n-3}{k}$. **(1 Mark)**

On the other hand, there are $\binom{n-1}{k-1}$ subsets of size k which contain a (since we then choose the remaining $k-1$ elements from $n-1$); similarly there are $\binom{n-2}{k-1}$ such subsets containing b but not a, and $\binom{n-3}{k-1}$ containing c but not a or b.

(2 Marks)

Thus $N = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$. **(1 Mark)**

iii) **(part a) done in lectures; rest unseen)**

a) There are $\binom{11}{4}$ such routes. **(1 Mark)**

This is because any shortest route from A to B consists of 11 steps altogether, of which 6 steps are to the right and 4 steps are up. The 4 steps up may be taken at any point, so there are $\binom{11}{4}$ possibilities. **(2 Marks)**

b) There are $\binom{6}{2}$ routes from A to C and $\binom{5}{3}$ routes from C to B, giving $\binom{6}{2}\binom{5}{3} = 150$ routes through C. **(2 Marks)**

c) If a route does not pass through D it must pass either through the grid point 2 units to the left of B (point E, say) or through the grid point 2 units down from B (point F, say). And from either E or F there is a unique shortest route to B. **(1 Mark)**

The number of routes via C and E is $\binom{6}{2}\binom{3}{2} = 45$. **(1 Mark)**

The number of routes via C and F is $\binom{6}{2}.1 = 15$. **(1 Mark)**

So the number of routes through C but not D is $45 + 15 = 60$. **(1 Mark)**

Solution to Question 2

ia) **(unseen)** No. Imagine the cells are alternately coloured black and white as on a chessboard. **(1 Mark)**

First note that the diagonally opposite corner cell is the same colour as that containing the prisoner, say white. **(1 Mark)**

In moving from adjacent cell to adjacent cell, the prisoner passes alternately through white and black squares. **(1 Mark)**

He needs to move through an even number of cells (64) and thus is forced to end up on a black one. **(1 Mark)**

iia) **(bookwork)** Suppose we have a finite set of items and properties $1, 2, 3, \dots, n$. Let $N(i_1, i_2, \dots, i_r)$ be the number of items which have the properties $i_1, i_2, \dots, i_r$ (and maybe others). Then the number of items with at least one of the properties is

$$\begin{aligned}
&N(1) + N(2) + N(3) + \cdots + N(n) \\
&- N(1,2) - N(1,3) - \cdots - N(n-1,n) \\
&+ N(1,2,3) + N(1,2,4) + \cdots \\
&- N(1,2,3,4) - \cdots \\
&\qquad\qquad \vdots \\
&+ (-1)^{n-1} N(1,2,3,\dots,n).
\end{aligned}$$

(3 Marks)

b) The number of perfect squares between 1 and 10,000 is 100 (since we have 1^2, 2^2, $\dots 100^2$). **(1 Mark)**

Similarly, the number of perfect cubes is 21. **(1 Mark)**

If a number is both a perfect square and a perfect cube then it is a 6-th power. The only ones between 1 and 10,000 are 1^6, 2^6, 3^6 and 4^6, so there are 4 of these. **(2 Marks)**

So, by the Inclusion/Exclusion Principle, the required number is

$$10,000 - (100 + 21 - 4) = 9883.$$

(1 Mark)

iiia) **(bookwork)** If you place more than n letters in n pigeon-holes then some pigeon-hole will contain more than one letter. **(2 Marks)**

b) **(standard problem)** Since all the numbers are even, any difference will also be even. So we need to show there is a difference divisible by 17. **(1 Mark)**
Label 17 pigeon-holes with the numbers $0, 1, \ldots, 16$. **(1 Mark)**
Place each of the 18 numbers in the pigeon-hole corresponding to its remainder on division by 17. **(1 Mark)**
There are 18 integers and 17 pigeon-holes, so by the Pigeon-Hole Principle there are two numbers in the same pigeon-hole. **(1 Mark)**
These two integers have the same remainder on division by 17, so their difference is divisible by 17. **(1 Mark)**

c) **(unseen)** All such sums are even and the largest possible sum of such a subset is $500 + 498 + 496 = 1494$. **(2 Marks)**
Label pigeon-holes with the even integers 2, 4, ..., 1494. So there are 747 pigeon-holes. **(1 Mark)**
Put each subset of size 3 into the pigeon-hole corresponding to its sum. There are $\binom{18}{3} = 816$ such subsets and 747 pigeon-holes, so by the Pigeon-Hole Principle there are two subsets in the same pigeon-hole. **(2 Marks)**
These two subsets have the same sum. **(1 Mark)**

Solution to Question 3

i) **(standard problem)**
There are many ways of calculating, using Theorem 42 (select a square) and Theorem 45 (disjoint boards) from the course. For example:
First use that the square bottom right is disjoint form the rest of B. Then choose the square two right and two down from top left, arriving at boards to which we can apply Theorem 45. **(2 Marks)**
We get:

$$r_B(x) = (1+x)\left((1+2x)r_C(x) + x(1+x)r_D(x)\right),$$

where C is the full 2×3 board and D is the full 2×2 board. **(3 Marks)**
So

$$\begin{aligned} r_B(x) &= (1+x)\left((1+2x)(1+6x+6x^2) + x(1+x)(1+4x+2x^2)\right) \\ &= 1 + 10x + 32x^2 + 41x^3 + 20x^4 + 2x^5 \,. \end{aligned}$$ (**3 Marks**)

[**Alternative:** Direct counting by hand will be given full marks if the correct answer is obtained; partial marks may be obtained for a partially correct answer depending on the explanation given for the counting procedure adopted.]

ii) **(bookwork)** We apply the IEP. The items are all $n!$ ways of placing n non-challenging rooks on the full $n \times n$ board. Property i is that the rook in row i is in B. **(3 Marks)**
So $N(i_1, i_2, \ldots, i_r)$ is the number of layouts of n non-challenging rooks on an $n \times n$ board such that the rooks in rows i_1, i_2, ..., i_r are in B. **(1 Mark)**
The number of ways of placing n non-challenging rooks on $\overline{B}$ is the number of the $n!$ items having none of the properties. **(1 Mark)**
This is $n!$ minus the number of items with at least one of the properties. **(1 Mark)**

By the IEP, this is

$$\begin{aligned}
&n! - N(1) - N(2) - N(3) - \cdots - N(n)\\
&+ N(1,2) + N(1,3) + \cdots + N(n-1,n)\\
&- N(1,2,3) - N(1,2,4) - \cdots\\
&+ N(1,2,3,4) + \cdots\\
&\qquad\vdots\\
&+ (-1)^n N(1,2,3,\ldots,n).
\end{aligned}$$

(2 Marks)

Now $N(i_1, i_2, \ldots, i_s)$ is the number of ways of placing s rooks in rows i_1, i_2, ..., i_s of B multiplied by $(n-s)!$, the number of ways of placing the remaining $n-s$ rooks. **(2 Marks)**

So the sum of all the $N(i_1, i_2, \ldots, i_s)$ terms is $r_s(n-s)!$. **(1 Mark)**

The required result comes from putting this into the formula above. **(1 Mark)**

iii) **(standard problem)** By parts (i) and (ii), the required number is

$$120 - 24.10 + 6.32 - 2.41 + 1.20 - 1.2 = 120 - 240 + 192 - 82 + 20 - 2 = 8.$$

(2 Marks)

[Follow through from the candidate's answer to (i).]

iv) **(unseen)** Any board which is the disjoint union of two 1×1 boards and a full 2×2 board.

(3 Marks)

Solution to Question 4

i) **(bookwork)** An $n \times n$ Latin square is an $n \times n$ matrix in which every one of the numbers $1, 2, \ldots n$ appears in each row and each column. **(2 Marks)**

ii) **(standard problem)** We want to extend a $p \times q$ Latin rectangle to an $n \times n$ Latin square, where $p = 3$, $q = 4$ and $n = 6$. This is possible iff $L(i) \geq 3 + 4 - 6 = 1$ for $1 \leq i \leq 6$, where $L(i)$ denotes the number of occurrences of i in the given rectangle. This happens iff $x = 3$. **(2 Marks)**

One extension is

$$\begin{pmatrix} 1 & 3 & 6 & 5 & 2 & 4\\ 4 & 2 & 1 & 6 & 3 & 5\\ 2 & 4 & 5 & 1 & 6 & 3\\ 3 & 1 & 2 & 4 & 5 & 6\\ 5 & 6 & 3 & 2 & 4 & 1\\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$$

(**6 Marks**)

[Marking: 4 marks for any correct extension to 3×6 or 6×4; 2 marks for any correct extension from there to 6×6. Partial marks available for "partially correct" extensions.]

iii) **(unseen problem)**

a) Suppose that $a_{ij} = a_{ij'}$. **(1 Mark)**
Then $2i + j - 2 \equiv 2i + j' - 2 \mod 7$. **(1 Mark)**
So $j \equiv j' \mod 7$ and since $1 \leq j, j' \leq 7$, we must have $j = j'$. **(1 Mark)**
Now suppose that $a_{ij} = a_{i'j}$. **(1 Mark)**
Then $3i + j - 2 \equiv 3i' + j - 2 \mod 7$. **(1 Mark)**
Thus $3i \equiv 3i' \mod 7$, and, since 3 and 7 are coprime, it follows that $i \equiv i' \mod 7$. Since $1 \leq i, i' \leq 7$, we must have $i = i'$. **(2 Marks)**
So A has all entries in each row different and all entries in each column different and so A is a Latin square. **(1 Mark)**

b) We need to show that each pair (i, j) with $1 \leq i, j \leq 7$ occurs exactly once among the pairs (a_{kl}, b_{kl}). **(2 Marks)**
Suppose that $(a_{kl}, b_{kl}) = (a_{mn}, b_{mn})$. **(1 Mark)**
Then $2k + l - 2 \equiv 2m + n - 2 \mod 7$ and $3k + l - 2 \equiv 3m + n - 2 \mod 7$. Subtracting, we find $k \equiv m \mod 7$ and then we deduce $l \equiv n \mod 7$. **(2 Marks)**
But $1 \leq k, l, m, n \leq 7$, so we must have $k = m$ and $l = n$. **(1 Mark)**
Thus the pairs are all distinct as required and A and B are orthogonal. **(1 Mark)**

Solution to Question 5

i) **(standard problem)**

a) The sum of the scores is $29 \neq 28 = \binom{8}{2}$, so this is impossible. **(1 Mark)**
b) By Landau's Theorem the scores are from a tournament iff their sum is $\binom{8}{2} = 28$ and for $1 \leq r \leq 7$ the sum of any r scores is at least $\binom{r}{2}$. **(2 Marks)**
But here, considering the last 4 scores, their sum is 5 and $5 < 6 = \binom{4}{2}$, so they are not the scores of a tournament. **(1 Mark)**
c) These are the scores of a tournament, by Landau's Theorem:
The total is $28 = \binom{8}{2}$. The sum of any r is at least the sum of the smallest r, shown in the table and this is at least $\binom{r}{2}$.

r	sum of r smallest scores	$\binom{r}{2}$
1	0	0
2	1	1
3	3	3
4	6	6
5	11	10
6	16	15
7	22	21

(4 Marks)

ii) **(unseen)**

a) A given number appears in the blocks corresponding to each other number in its row and in its column, hence it appears in 6 blocks. **(2 Marks)**
b) Let x, y be a pair of numbers.

First consider the case where x and y are in the same row. They appear together in exactly 2 blocks, namely those corresponding to the other two entries in their row. **(2 Marks)**
Similarly if x and y are in the same column. **(1 Mark)**
Now suppose x and y are in different rows and different columns. Then they appear together in exactly two blocks, namely the one corresponding to the square at the intersection of the row of x and the column of y and the one corresponding to the square at the intersection of the row of y and the column of x. **(3 Marks)**
c) Clearly, we have 16 varieties, the numbers $1, 2, \ldots, 16$. And we have 16 blocks, one for each square. **(1 Mark)**
Each block contains 6 varieties and each variety appears in 6 blocks. **(1 Mark)**
Each pair of varieties appears in precisely 2 blocks. Thus we have a $(16, 16, 6, 6, 2)$ design. **(1 Mark)**
d) For each block make a complement block by taking the complement in the set of all varieties $\{1, 2, \ldots, 16\}$. **(1 Mark)**
Each complement block contains $16 - 6 = 10$ varieties. **(1 Mark)**
Each variety was in 6 blocks, hence absent from 10 and thus is in 10 complement blocks. **(1 Mark)**
Consider the number of complement blocks containing a given pair of varieties x, y. This is the number of original blocks containing neither x nor y. Since each of x and y was in 6 blocks, and they were both together in 2 blocks, this is $16 - (6 + 6 - 2) = 16 - 10 = 6$. **(2 Marks)**
Hence the complement blocks are the blocks of a $(16, 16, 10, 10, 6)$ design.
(1 Mark)