MAS334 COMBINATORICS — AUTUMN SEMESTER 2016-2017
EXAM SOLUTIONS AND MARK SCHEME

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Solution to Question 1

(ia) **(bookwork)**

There are $\binom{n-1}{k-1}$ solutions. **(1 Mark)**

This can be seen by noting that the solutions are in bijection with the ways of inserting $k-1$ dividers into the $n-1$ spaces between n items in a line.

(2 Marks)

(ib) **(unseen, similar to seen problem)**

Let $y_1 = y_1' + 1$, $y_2 = y_2' + 2$, $y_3 = y_3' + 3$ and $y_4 = y_4' + 4$. **(2 Marks)**

Then the required solutions are in bijection with positive integer solutions to

$$y_1' + y_2' + y_3' + y_4' = 37 - (1 + 2 + 3 + 4) = 27, \qquad (\textbf{2 Marks})$$

By part (a), the number of solutions is $\binom{26}{3} = 2600$. **(1 Mark)**

(ii) **(unseen, similar to seen problems)**

(a) There are $\binom{6}{2}\binom{3}{1} = 15.3 = 45$ such routes. **(2 Marks)**

(b) $\binom{9}{3} - 45 = 84 - 45 = 39$. **(2 Marks)**

(c)

$$\binom{m+n}{n} - \binom{i+j}{i}\binom{m+n-i-j}{m-i}.$$

(3 Marks)

(iii) **(unseen, similar to seen problems)**

(a) On the one hand, there are $\binom{n+1}{a+b+1}$ ways to choose $a + b + 1$ numbers from the set $\{0, 1, \ldots, n\}$. **(1 Mark)**

On the other hand, we consider the number k in position $a+1$ when we write our chosen $a + b + 1$ numbers in increasing order. **(1 Mark)**

Then the first a of the chosen numbers are chosen from $\{0, 1, \ldots, k-1\}$ and the last b from $\{k+1, \ldots, n-1, n\}$. This can be done in $\binom{k}{a}\binom{n-k}{b}$ ways. Thus the total number of ways is $\sum_{k=0}^{n} \binom{k}{a}\binom{n-k}{b}$. **(3 Marks)**

Thus we have the required identity. **(1 Mark)**

(b) Using (a) twice,

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{j}{a}\binom{k}{b}\binom{n-j-k}{c} &= \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{j}{a}\binom{(n-k)-j}{c}\binom{k}{b}\\ &= \sum_{k=0}^{n}\binom{n+1-k}{a+c+1}\binom{k}{b} \qquad (\textbf{2 Marks})\\ &= \sum_{k=0}^{n+1}\binom{n+1-k}{a+c+1}\binom{k}{b}\\ &= \binom{n+2}{a+b+c+2}. \qquad (\textbf{2 Marks})\end{aligned}$$

[**Alternative:** Counting argument, as in part (a): choose $a+b+c+2$ numbers from the set $\{0,1,\dots,n+1\}$ and consider the number j in position $a+1$ and the number k such that $j+k+1$ is in position $a+b+2$ when we write our chosen $a+b+c+2$ numbers in increasing order. Then the first a of the chosen numbers are chosen from $\{0,1,\dots,j-1\}$, the next b from $\{j+1,\dots,j+k\}$ and the final c from $\{j+k+2,\dots,n+1\}$.]

Solution to Question 2

(ia) **(unseen, similar to seen problems)**

This is possible if and only if m or n is even. **(1 Mark)**
If n is even, fill each row completely and so the whole board. If m is even, fill each column completely and so the whole board. **(2 Marks)**
If both m and n are odd, the total number of squares mn is also odd, but dominoes cover an even number of squares so this is impossible. **(1 Mark)**

(ib) **(unseen, similar to seen problems)**

Consider shading the squares alternately black and white. **(1 Mark)**

For the whole rectangle, we have the same number of black and white squares, because we have the same number in each column. When we remove the corners, we remove two squares of the same colour, so leaving different numbers of black and white squares. Since each domino covers a square of each colour, it is impossible to cover the modified board. **(3 Marks)**

(iiia) **(unseen, standard problem)**

Label PHs $0, 1, 2, \ldots, 123456788$ corresponding to the possible remainders on division by 123456789. **(2 Marks)**

Assign the first 123456790 powers of 17 to the PH corresponding to their remainder on division by 123456789. **(1 Mark)**
Since there are more items than PHs, by the PHP there are two in the same PH, that is two powers of 17 with the same remainder on division by 123456789. **(1 Mark)**
The difference between these two powers of 17 is divisible by 123456789. **(1 Mark)**

(iiib) **(unseen, elementary)**

Suppose instead that each box contains at most $\lfloor n/k \rfloor$ items. Then the total number of items is at most $k\lfloor n/k \rfloor \leq kn/k = n$, contradicting that we had $n+1$ items in total. **(3 Marks)**

(iva) **(bookwork)** Suppose we have a finite set of items and properties $1, 2, 3, \ldots, n$. Let $N(i_1, i_2, \ldots, i_r)$ be the number of items which have the properties $i_1, i_2, \ldots, i_r$ (and maybe others). Then the number of items with at least one of the properties is

$$\begin{aligned}
&N(1) + N(2) + N(3) + \cdots + N(n) \\
&- N(1,2) - N(1,3) - \cdots - N(n-1,n) \\
&+ N(1,2,3) + N(1,2,4) + \cdots \\
&- N(1,2,3,4) - \cdots \\
&\qquad\qquad \vdots \\
&+ (-1)^{n-1} N(1,2,3,\ldots,n).
\end{aligned}$$

(3 Marks)

ivb) **(unseen, similar to homework problems)**

Let P_1, P_2, P_3 be the property that a permutation fixes 8, fixes 9, fixes 10 respectively. **(1 Mark)**

We have $N(i) = 9!$ for $i = 1, 2, 3$, $N(i, j) = 8!$ for all i, j, $N(i, j, k) = 7!$ for all i, j, k. **(2 Marks)**

We want the number of items with at least one of the properties. **(1 Mark)**

By the Inclusion/Exclusion Principle, this is:

$$3 \cdot 9! - 3 \cdot 8! + 7! = 972720.$$ (**2 Marks**)

Solution to Question 3

(i) **(unseen, standard)**

There are many ways of calculating, using Theorem 43 (select a square) and Theorem 46 (disjoint boards) from the course. The obvious one for this example is to select the third square on the second row down, to obtain disjoint boards at the next stage.

We get:

$$r_B(x) = r_C(x) + xr_D(x),$$

where C consists of two disjoint boards, a full 2×1 and a 2×3 board with one corner square deleted and D consists of disjoint full 1×1 and 2×2 boards.

(3 Marks)

So

$$\begin{aligned} r_B(x) &= (1+2x)(1+5x+4x^2) + x(1+x)(1+4x+2x^2) \\ &= 1 + 8x + 19x^2 + 14x^3 + 2x^4 . \qquad (\textbf{3 Marks}) \end{aligned}$$

[**Alternative:** Direct counting by hand will be given full marks if the correct answer is obtained (even without justification); partial marks may be obtained for a partially correct answer depending on the explanation given for the counting procedure adopted.]

(iii) **($m = n = 4$ case done in lectures)**

Fix $1 \le k \le n$ and consider the number of ways of placing k non-challenging rooks on the given board. There are $\binom{m}{k}$ ways to choose k columns for the k rooks and $\binom{n}{k}$ ways to choose k rows for the k rooks. **(2 Marks)**

Within the chosen columns and rows, there are k possible positions for the rook on the first row, then $k-1$ on the second row and $k-2$ on the third row and so on. **(2 Marks)**

Altogether this gives us

$$\binom{m}{k}\binom{n}{k} k.(k-1).(k-2)\dots 1$$

ways to place k non-challenging rooks. So the rook polynomial is as claimed.

(1 Mark)

(iii) **(unseen)**

(a) Yes. This is the rook polynomial of any board made up of disjoint copies of a full 1×1 board and two full 2×2 boards. **(2 Marks)**

(b) No. The coefficient of x is 4 and this is the number of unshaded squares. So there are at most $\binom{4}{2} = 6$ ways to place two rooks. But the coefficient of x^2 is 7 and this is the number of ways of placing 2 non-challenging rooks. So it's impossible. **(2 Marks)**

(iva) **(unseen, standard)**

The sum of the scores is $\binom{n}{2}$. And, for each r, the sum of the least r scores is $\sum_{i=0}^{r-1} i = \binom{r}{2}$. So, by Landau's Theorem, these are the scores of a tournament of n players. **(3 Marks)**

[**Alternatively**, exhibit a suitable tournament: p_i beats p_j for $i > j$.]

(ivb) **(on problem sheet)** The sum of the scores is $nm = (2m+1)m = \binom{n}{2}$. For $1 \leq r < n$, the sum of any r scores is $rm = r(n-1)/2 \geq r(r-1)/2 = \binom{r}{2}$. So, by Landau's Theorem, these are the scores of a tournament of n players.

(2 Marks)

(ivc) **(unseen)** For the first set of scores, combine a tournament A from the first part with a tournament B from the second part, with all the A players beating all the B players. Thus we add $n = 2m+1$ to every A score, giving the required scores. **(2 Marks)**

For the second set of scores, combine a tournament A from the first part with a tournament B from the second part, with all the B players beating all the A players. Thus we add $n = 2m+1$ to every B score, giving the required scores.

(2 Marks)

Solution to Question 4

(i) **(standard problem)** We want to extend a $p \times q$ Latin rectangle to an $n \times n$ Latin square, where $p = 4$, $q = 4$ and $n = 6$. This is possible iff $L(i) \geq 4 + 4 - 6 = 2$ for $1 \leq i \leq 6$, where $L(i)$ denotes the number of occurrences of i in the given rectangle. This happens iff $x = 1$. **(2 Marks)**
One extension is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 1 & 6 & 2 & 3 & 5 \\ 5 & 6 & 1 & 3 & 2 & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{pmatrix} \qquad (\textbf{4 Marks})$$

[Marking: 2 marks for any correct extension to 4×6 or 6×4; 2 marks for any correct extension from there to 6×6. Partial marks available for "partially correct" extensions.]

(iia) **(bookwork)** Two $n \times n$ Latin squares $L = (l_{ij})$, $M = (m_{ij})$ are called *orthogonal* if the pairs (l_{ij}, m_{ij}) include all the possibilities $(1, 1), (1, 2), \ldots, (n, n)$. **(2 Marks)**

(iib) **(unseen problem; used on previous exams, last in 2013-14)** Suppose that $(A_k)_{ij} = (A_k)_{i'j'}$ and $(A_h)_{ij} = (A_h)_{i'j'}$ for $k \neq h$. **(2 Marks)**
Then $ki + j \equiv ki' + j' \bmod p$ and $hi + j \equiv hi' + j' \bmod p$. **(1 Mark)**
Thus $(k-h)i \equiv (k-h)i'$ and since p does not divide $k-h$, this gives $i \equiv i' \bmod p$. Since $1 \leq i, i' \leq p$, it follows that $i = i'$. Then $j \equiv j' \bmod p$ and so $j = j'$. **(2 Marks)**
So there are no duplicates among the pairs $((A_k)_{ij}, (A_h)_{ij})$ and the squares A_k and A_h are orthogonal. **(1 Mark)**

(iii) **(bookwork)** Let x be one of the varieties and assume x is in r_x blocks. **(1 Mark)**
Consider all the blocks and count the number of *pairs* that involve x. **(1 Mark)**
Firstly, since x can be paired with $k - 1$ others in each of its blocks, this total is $r_x(k - 1)$. **(1 Mark)**
On the other hand, there are $v - 1$ varieties apart from x. Each pair of x with one other appears λ times in the design. So the total is $\lambda(v - 1)$. **(1 Mark)**
So $r_x(k - 1) = \lambda(v - 1)$ and thus $r_x = \frac{\lambda(v-1)}{k-1}$. **(1 Mark)**
This is independent of x and we call it r. Now, since each variety occurs in r blocks, the total number of entries in all blocks is vr. **(1 Mark)**
On the other hand, we have b blocks of k entries, so this is also bk. So $vr = bk$ and thus $r = \frac{bk}{v}$. **(1 Mark)**

(iv) **(unseen)**

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad (\textbf{1 Mark})$$

Thus

$$M^T M = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \qquad (\textbf{1 Mark})$$

By a theorem from the course, this form of $M^T M$ shows that these are the blocks of a design, with $r = 3$ and $\lambda = 2$. **(1 Mark)**

The parameters of the design are $(v, b, r, k, \lambda) = (4, 4, 3, 3, 2)$. **(1 Mark)**