The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

Vector spaces and Fourier theory (SOLUTIONS) **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

{B} means bookwork, {S} means seen similar, {U} means unseen

1 (i) **{B}**

(a) If $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then $\lambda_1 = \cdots = \lambda_n = 0$.

(b) Every vector $v \in V$ can be expressed as $\lambda_1 v_1 + \cdots + \lambda_n v_n$, for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

(c) The vectors must both be linearly independent and span.

(6 marks)

(ii) **{S}**

(a) This is not a subspace; the zero vector $(0,0,0)^T$ is not in U_1.

(b) This is not a subspace; the zero vector $(0,0,0)^T$ is not in U_2.

(c) This is not a subspace; it is not closed under scalar multiplication. E.g., $(0,0,1) \in U_3$, but $(-1)(0,0,1) = (0,0,-1) \notin U_3$.

(d) This is a subspace: it contains $(0,0,0)^T$ and it is closed under addition and scalar multiplication: if $v = (x,y,z)^T, v' = (x',y',z')^T \in U_4$ and $\alpha, \beta \in \mathbb{R}$, then

$$\alpha x + \beta x' + 3(\alpha y + \beta y') + \alpha z + \beta z' = \alpha(x + 3y + z) + \beta(x' + 3y' + z') = 0,$$

so $\alpha v + \beta v' \in U_4$.

(7 marks)

(iii) **{B}**

$$U + W = \{v \in V \mid v = u + w \text{ for some } u \in U,\ w \in W\}.$$

As U is a subspace we have $0_V \in U$, and as W is a subspace we have $0_V \in W$, so $0_V = 0_V + 0_V \in U + W$.

Next, suppose we have $v, v' \in U + W$, $\alpha, \alpha' \in \mathbb{R}$.

Then $v = u + w$, $v' = u' + w'$ with $u, u' \in U$ and $w, w' \in W$.

So

$$\alpha v + \alpha' v' = (\alpha u + \alpha' u') + (\alpha w + \alpha' w') \in U + W,$$

since $\alpha u + \alpha' u' \in U$ (as U is a subspace) and $\alpha w + \alpha' w' \in W$ (as W is a subspace).

So $U + W$ is a subspace. ***(7 marks)***

(iv) **{S}** Suppose that $(x,y,z)^T \in U \cap W$. Then $(x,y,z)^T = (a, 2a, a)^T$ for some $a \in \mathbb{R}$, since $(x,y,z)^T \in U$ and $a = 2a$ since $(x,y,z)^T \in W$. Thus $a = 0$ and $(x,y,z)^T = (0,0,0)^T$, so $U \cap W = \{0_V\}$.

Now let $(x,y,z)^T \in \mathbb{R}^3$. Then we have

$$(x,y,z)^T = (y - x, 2(y - x), y - x) + (2x - y, 2x - y, z - y + x) \in U + W,$$

so $U + W = \mathbb{R}^3$. ***(5 marks)***

2 (i) **{B}** Suppose that $\{v_1, \ldots, v_r\}$ is a linearly independent set in the vector space V, and $\{w_1, \ldots, w_s\}$ spans V. Then $r \leq s$. ***(2 marks)***

(ii) **{U}**

(a) True (by Steinitz).

(b) False; e.g., $V = \mathbb{R}^2$, $v_1 = v_2 = (1,0)^T$.

(c) True (by Steinitz).

(d) False; e.g., $V = \mathbb{R}^2$, $v_1 = v_2 = (1,0)^T$. ***(6 marks)***

(iii) **{B}** $\ker(\phi) = \{v \in V \mid \phi(v) = 0_W\}$.

Suppose that ϕ is injective, so whenever $\phi(v) = \phi(v')$ we have $v = v'$. Suppose that $v \in \ker(\phi)$. Then $\phi(v) = 0_W = \phi(0_V)$. As ϕ is injective and $\phi(v) = \phi(0_V)$, we must have $v = 0_V$. Thus $\ker(\phi) = \{0_V\}$, as claimed.

Conversely, suppose that $\ker(\phi) = \{0_V\}$. Suppose that $\phi(v) = \phi(v')$. Then $\phi(v - v') = \phi(v) - \phi(v') = 0$, so $v - v' \in \ker(\phi) = \{0_V\}$, so $v - v' = 0_V$, so $v = v'$. This means that ϕ is injective. ***(7 marks)***

(iv) **{B}** $\dim(U) + \dim(W) = \dim(U \cap W) + \dim(U + W)$. ***(1 mark)***

(v) **{S}**

(a) $U = \left\{ \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} \mid b, c, f \in \mathbb{R} \right\}$, and $\dim(U) = 3$.

$W = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \mid b, c, e, f, h, i \in \mathbb{R} \right\}$, and $\dim(W) = 6$.

Then $U \cap W = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & -f & 0 \end{pmatrix} \mid f \in \mathbb{R} \right\}$, and $\dim(U \cap W) = 1$.

So $\dim(U + W) = 3 + 6 - 1 = 8$. ***(8 marks)***

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin U + W$. ***(1 mark)***

3 (i) **{B}** The standard basis is: $1, x, x^2, x^3$.

Since this consists of 4 polynomials, the dimension of V is 4. ***(3 marks)***

(ii) **{S}** Let $f(x) = a + bx + cx^2 + dx^3 \in V$. Then

$$f(x) = (a+b+c+d).1 + (b+c+d)(x-1) + (c+d)(x^2-x) + d(x^3-x^2),$$

so the given vectors span V.

Also

$$\lambda_1 + \lambda_2(x-1) + \lambda_3(x^2-x) + \lambda_4(x^3-x^2) = 0$$
$$\Leftrightarrow \lambda_1 - \lambda_2 + (\lambda_2-\lambda_3)x + (\lambda_3-\lambda_4)x^2 + \lambda_4 x^3 = 0.$$

Equating coefficients, we see that $\lambda_4 = \lambda_3 = \lambda_2 = \lambda_1 = 0$, so these vectors are also linearly independent. Hence they are a basis. ***(4 marks)***

[Alternatively, check one of linear independence or spanning, and then note that you have the correct number of vectors, so it must be a basis.]

(iii) **{S}** Let $f, g \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}\phi(\alpha f + \beta g) &= (1+x)(\alpha f + \beta g)'(x)\\ &= (1+x)(\alpha f'(x) + \beta g'(x))\\ &= \alpha(1+x)f'(x) + \beta(1+x)g'(x)\\ &= \alpha\phi(f) + \beta\phi(g).\end{aligned}$$

So ϕ is linear. ***(3 marks)***

(iv) **{S}** Find the matrix of ϕ with respect to the standard basis. By calculating ϕ on each of the standard basis vectors, we find the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 2 & 3\\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(3 marks)

 Question 3 continued on next page

3 (continued)

(v) **{S}** By calculating ϕ on each of the given basis vectors and writing the answer in terms of them, we find the matrix

$$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

[Alternatively, calculate with the change of basis matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.]$$

(5 marks)

(vi) **{S}** The kernel is the set of constant polynomials. ***(2 marks)***

(vii) **{U}**

We see that $\mathrm{im}(\phi) = \{b + (b + 2c)x + (2c + 3d)x^2 + 3dx^3 \mid b, c, d \in \mathbb{R}\}$.

Suppose that $f \in \mathrm{im}(\phi)$, so $f(x) = b + (b + 2c)x + (2c + 3d)x^2 + 3dx^3$ for some $b, c, d \in \mathbb{R}$. Then $f(-1) = b - (b + 2c) + (2c + 3d) - 3d = 0$. So $\mathrm{im}(\phi) \subseteq \{f \in V \mid f(-1) = 0\}$.

[Alternatively, every element of $\mathrm{im}(\phi)$ is divisible by $1+x$ and so has a root at $x = -1$.]

On the other hand, if $f(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$ satisfies $f(-1) = 0$, then $f(x) = b + (b + 2c)x + (2c + 3d)x^2 + 3dx^3$ where $b = \alpha$, $c = \dfrac{\beta - \alpha}{2}$ and $d = \dfrac{\gamma - \beta + \alpha}{3}$. So $\{f \in V \mid f(-1) = 0\} \subseteq \mathrm{im}(\phi)$.

[Alternatively, argue via dimensions.]

(5 marks)

4 (i) (a) **{S}** Let $u_1 = x, u_2 = x^2, u_3 = x^3$. Then $\langle u_1, u_2\rangle = \langle u_2, u_3\rangle = 0$ and $\langle u_1, u_1\rangle = \int_{-1}^{1} x^2\,dx = 2/3$, $\langle u_1, u_3\rangle = \int_{-1}^{1} x^4\,dx = 2/5$.

So the Gram-Schmidt procedure gives

$$\begin{aligned} v_1 &= u_1, \\ v_2 &= u_2, \\ v_3 &= u_3 - \frac{\langle u_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - 0 = x^3 - \frac{3}{5}x. \end{aligned}$$

(6 marks)

(b) **{S}** The closest element to 1 is $\pi(1)$ where $\pi : \mathbb{R}[x] \to U$ is orthogonal projection onto U.

We have

$$\begin{aligned} \pi(1) &= \sum_{i=1}^{3} \frac{\langle 1, v_i\rangle}{\langle v_i, v_i\rangle} v_i \\ &= 0 + \frac{2/3}{2/5}x^2 + 0 = \frac{5}{3}x^2. \end{aligned}$$

So the closest element is $\frac{5}{3}x^2$. ***(6 marks)***

(c) **{S}** The distance is $\|1 - \frac{5}{3}x^2\| = \sqrt{\langle 1 - \frac{5}{3}x^2, 1 - \frac{5}{3}x^2\rangle}$.

Since

$$\begin{aligned} \int_{-1}^{1} (1 - \frac{5}{3}x^2)^2\,dx &= \int_{-1}^{1} (1 - \frac{10}{3}x^2 + \frac{25}{9}x^4)\,dx \\ &= \left[x - \frac{10}{9}x^3 + \frac{5}{9}x^5\right]_{-1}^{1} = 2 - \frac{20}{9} + \frac{10}{9} = \frac{8}{9}, \end{aligned}$$

the distance is $\sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2}$.

[Follow-through from the candidate's answer to (b) where appropriate.] ***(5 marks)***

4 (continued)

(ii) (a) **{B}** For $v, w \in V$, $|\langle v, w\rangle| \leq \|v\|\,\|w\|$, with equality if and only if v and w are linearly dependent. ***(3 marks)***

(b) **{S}** Take $f(t) = \sqrt{\sin t}$ and $g(t) = 1$ in the Cauchy-Schwarz inequality.

Then $\langle f, g\rangle = \displaystyle\int_0^\pi \sqrt{\sin t}\, dt$.

And

$$\begin{aligned}\|f\|\,\|g\| = \sqrt{\langle f, f\rangle\langle g, g\rangle} &= \sqrt{\int_0^\pi \sin t\, dt \int_0^\pi 1\, dt} \\ &= \sqrt{[-\cos t]_0^\pi\,[t]_0^\pi} \\ &= \sqrt{2\pi},\end{aligned}$$

so the Cauchy-Schwarz inequality gives the result. ***(5 marks)***

5 (i) **{B/S}** The inner product is given by $\langle X, Y\rangle = \text{trace}(XY^T)$.
The distance is

$$\|A-B\| = \sqrt{\langle A-B, A-B\rangle} = \sqrt{\text{trace}((A-B)(A-B)^T)}$$
$$= \sqrt{\text{trace}\left(\begin{pmatrix}2 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\right)} = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3.$$

(5 marks)

(ii) (a) **{B}** It is orthogonal (standard Fourier theory), but not orthonormal since, for example, $\langle 1, 1\rangle = 2\pi$. ***(2 marks)***

(b) **{S}** We have $\cos 2t \cos t = (\cos 3t + \cos t)/2$ and $\cos 4t \cos t = (\cos 5t + \cos 3t)/2$.
Then

$$\langle \cos 2t \cos t, \cos 4t \cos t\rangle = \langle \cos 3t, \cos 3t\rangle/4 = \pi/4,$$

and

$$\langle \cos 2t \cos t, \cos 2t \cos t\rangle = \langle \cos 4t \cos t, \cos 4t \cos t\rangle = \pi/2,$$

so that the cosine of the angle between the two functions is $\dfrac{\pi/4}{\sqrt{\pi/2.\pi/2}} = 1/2$, so the angle is $\pi/3$. ***(8 marks)***

(iii) (a) **{S}** We have

$$D(a + b\cos 3t + c\sin 3t) = 0 + 3c\cos 3t - 3b\sin 3t,$$

so the matrix is $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 3\\ 0 & -3 & 0\end{pmatrix}$. ***(3 marks)***

(b) **{S}** We calculate

$$\begin{aligned}&\langle D(a + b\cos 3t + c\sin 3t), \alpha + \beta\cos 3t + \gamma\sin 3t\rangle\\ &\qquad = \langle -3b\sin 3t + 3c\cos 3t, \alpha + \beta\cos 3t + \gamma\sin 3t\rangle\\ &\qquad = (-3b\gamma + 3c\beta)\pi.\end{aligned}$$

We need $\hat{D}$ to be defined so that

$$\begin{aligned}&\langle D(a + b\cos 3t + c\sin 3t), \alpha + \beta\cos 3t + \gamma\sin 3t\rangle\\ &\qquad = \langle a + b\cos 3t + c\sin 3t, \hat{D}(\alpha + \beta\cos 3t + \gamma\sin 3t)\rangle,\end{aligned}$$

and so we need $\hat{D}(\alpha + \beta\cos 3t + \gamma\sin 3t) = -3\gamma\cos 3t + 3\beta\sin 3t$, which is $-D(\alpha + \beta\cos 3t + \gamma\sin 3t)$. Therefore $\hat{D} = -D$.
(7 marks)

End of Question Paper