The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2009–2010**

Vector spaces and Fourier theory (SOLUTIONS) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

{B} means bookwork, {S} means seen similar, {U} means unseen

1 (i) **{B}**

(a) If $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then $\lambda_1 = \cdots = \lambda_n = 0$.

(b) Every vector $v \in V$ can be expressed as $\lambda_1 v_1 + \cdots + \lambda_n v_n$, for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

(c) The vectors must both be linearly independent and span.

(6 marks)

(ii) **{S}**

(a) The zero vector $(0,0,0)^T$ is not in U_1.

(b) This is not closed under scalar multiplication; e.g., $(0,0,1) \in U_2$, but $(-1)(0,0,1) = (0,0,-1) \notin U_2$.

(c) This is not closed under addition or scalar multiplication, e.g., $(1,0,0) \in U_3$, but $2(1,0,0) = (1,0,0) + (1,0,0) = (2,0,0) \notin U_3$.

(6 marks)

(iii) **{B}** As U is a subspace we have $0 \in U$, and as W is a subspace we have $0 \in W$, so $0 \in U \cap W$. Next, suppose we have $v, v' \in U \cap W$. Then $v, v' \in U$ and U is a subspace, so $v + v' \in U$. Similarly, we have $v, v' \in W$ and W is a subspace so $v + v' \in W$. This shows that $v + v' \in U \cap W$, so $U \cap W$ is closed under addition. Finally consider $v \in U \cap W$ and $\alpha \in \mathbb{R}$. Then $v \in U$ and U is a subspace so $\alpha v \in U$. Similarly $v \in W$ and W is a subspace so $\alpha v \in W$. This shows that $\alpha v \in U \cap W$, so $U \cap W$ is closed under scalar multiplication, and $U \cap W$ is a subspace. ***(5 marks)***

(iv) (a) **{B}** $U + W = \{v \in V \mid v = u + w \text{ for some } u \in U,\ w \in W\}$.

We have $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

(3 marks)

(b) **{U}** We know $5 = \dim U \leq \dim(U + W) \leq \dim V = 7$, so the relation above becomes $2 \leq \dim(U \cap W) \leq 4$. ***(2 marks)***

(v) **{S}** If $(x,y,z)^T \in U \cap W$, then $x = 0$ and $x = y = z$, so $(x,y,z)^T = (0,0,0)^T$, as required. Also, if $(a,b,c)^T \in \mathbb{R}^3$, then

$$(a,b,c)^T = (a,a,a)^T + (0, b-a, c-a)^T,$$

so $(a,b,c)^T \in U + W$ as required. ***(3 marks)***

2 (i) **{S}** Suppose $\alpha f + \beta g + \gamma h = 0$. Then for every value of x, we must have $\alpha f(x) + \beta g(x) + \gamma h(x) = 0$. Put $x = 0$; as $f(0) = h(0) = 0$ and $g(0) = 1$, we get $\beta = 0$. Now when $x = \pi$, we have $f(\pi) = 0$ and $h(\pi) = \pi^2$. So $\gamma = 0$. Finally, choose, e.g., $x = \pi/2$ to get $\alpha = 0$. ***(4 marks)***

(ii) **{B}** Since $\mathcal{V}$ is a basis, it spans, and so v certainly can be expressed as a linear combination. Suppose there are two such representations:

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n = \mu_1 v_1 + \cdots + \mu_n v_n;$$

subtracting gives $(\lambda_1 - \mu_1)v_1 + \cdots + (\lambda_n - \mu_n)v_n = 0$, a linear relationship between a linearly independent set; thus the coefficients vanish, and each $\lambda_i = \mu_i$, and the two representations are the same. ***(4 marks)***

(iii) **{B}** Suppose that $\{v_1, \ldots, v_r\}$ is a linearly independent set in the vector space V, and $\{w_1, \ldots, w_s\}$ spans. Then $r \leq s$. ***(2 marks)***

(iv) **{U}**

(a) True (by Steinitz).

(b) False; e.g., $V = \mathbb{R}^2$, $v_1 = v_2 = (1,0)^T$;

(c) True (by Steinitz).

(d) False; e.g., $V = \mathbb{R}^2$, $v_1 = v_2 = (1,0)^T$. ***(6 marks)***

(v) (a) **{U}** Linearly independent: if

$$\lambda_1 v_1 + \lambda_2(v_2 - v_1) + \lambda_3(v_3 - v_2) + \lambda_4(v_4 - v_3) = 0,$$

then $(\lambda_1 - \lambda_2)v_1 + (\lambda_2 - \lambda_3)v_2 + (\lambda_3 - \lambda_4)v_3 + \lambda_4 v_4 = 0$, then $\lambda_1 = \cdots = \lambda_4 = 0$, since the coefficients all vanish.

Spans: given $v \in V$, we can write

$$\begin{aligned} v &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = (a_1 + a_2 + a_3 + a_4)v_1 + \\ &\quad (a_2 + a_3 + a_4)(v_2 - v_1) + (a_3 + a_4)(v_3 - v_2) + a_4(v_4 - v_3). \end{aligned}$$

(3 marks)

(b) **{S}** As in the last part:

$$\begin{aligned} f &= af_0 + bf_1 + cf_2 + df_3 = (a + b + c + d)f_0 + \\ &\quad (b + c + d)(f_1 - f_0) + (c + d)(f_2 - f_1) + d(f_3 - f_2). \end{aligned}$$

and so $\alpha = a + b + c + d$, $\beta = b + c + d$, $\gamma = c + d$, $\delta = d$, and

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

(4 marks)

(vi) **{S}** If $a + bx + cx^2 + dx^3 \in V$, then $f''(x) = 2c + 6dx$ and so $2c + 6d = 0$. Also $f(0) = a = 0$ and we find the general element $bx - 3dx^2 + dx^3$, with basis $\{x, x^3 - 3x^2\}$. ***(2 marks)***

3 (i) **{B}** ϕ is *linear* if: for any v and v' in V, we have $\phi(v+v') = \phi(v) + \phi(v')$ in W and also for any $\alpha \in \mathbb{R}$ and $v \in V$ we have $\phi(\alpha v) = \alpha\phi(v)$ in W.

(3 marks)

(ii) **{B}** The kernel is the set $\{v \in V \mid \phi(v) = 0\}$. The image is the set $\{w \in W \mid w = \phi(v) \text{ for some } v \in V\}$.

Firstly, $\phi(0_V) = 0_W$, so $0_W \in \mathrm{im}(\phi)$. Now suppose we have $w, w' \in \mathrm{im}(\phi)$. This means that we can find $v, v' \in V$ with $\phi(v) = w$ and $\phi(v') = w'$. We thus have $v + v', \alpha v \in V$ and as ϕ is linear we have $\phi(v+v') = \phi(v) + \phi(v') = w + w'$ and $\phi(\alpha v) = \alpha\phi(v) = \alpha w$. This shows that $w + w'$ and αw lie in $\mathrm{im}(\phi)$.

(6 marks)

(iii) **{S}**

We have $D(e^x) = e^x$; $D(xe^x) = e^x + xe^x$; $D(x^2e^x) = 2xe^x + x^2e^x$, so $D(ae^x + bxe^x + cx^2e^x) = (a+b)e^x + (b+2c)xe^x + cx^2e^x$, and if $f \in V$, we see that $D(f) \in V$.

Differentiation is always linear as $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$.

The matrix is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$; from this we see that the only eigenvalue is 1. The formula above shows that $D(f) = f$ precisely when $a = a + b$, $b = b + 2c$ and $c = c$; these give $b = c = 0$, so the only eigenvector is $f(x) = e^x$.

(8 marks)

(iv) **{S}** Note that $\phi(a + bx + cx^2 + dx^3) = (b, -b-c-d, -b-5c-19d, b+6c+27d)^T$, and $\phi(f) = 0$ implies that f is a constant. So $\ker\phi$ is spanned by 1. We put $f_4 = 1$, and extend to a basis for $\mathbb{R}[x]_{\leq 3}$ by adding e.g., $f_1 = x$, $f_2 = x^2$ and $f_3 = x^3$. Then put $v_1 = \phi(f_1) = (1, -1, -1, 1)^T$, $v_2 = \phi(f_2) = (0, -1, -5, 6)^T$, $v_3 = \phi(f_3) = (0, -1, -19, 27)^T$, and extend to a basis of $\mathbb{R}^4$ by adding e.g., $v_4 = (1, -1, -1, 0)^T$ [if $v_4 \in \mathrm{im}(\phi)$, then $c + d = 0$, $5c + 19d = 0$ but $6c + 27d = -1$; these are inconsistent, so $v_4 \notin \mathrm{im}(\phi)$, and v_4 is independent of v_1, v_2, v_3.]

(8 marks)

4 (i) **{B}** An *inner product* on V is a rule that gives a number $\langle u, v\rangle \in \mathbb{R}$ for each $u, v \in V$, with the following properties:

(a) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.

(b) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ for all $u, v \in V$ and $\alpha \in \mathbb{R}$.

(c) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.

(d) We have $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ if and only if $u = 0$.

(5 marks)

(ii) (a) **{B}** We require $\langle v_i, v_j\rangle = 0$ if $i \neq j$. ***(1 mark)***

(b) **{B}** Suppose we have a linear relationship $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. Then

$$\langle \lambda_1 v_1 + \cdots + \lambda_n v_n, v_i\rangle = \lambda_1\langle v_1, v_i\rangle + \cdots + \lambda_n\langle v_n, v_i\rangle = 0,$$

and using the orthogonality relations, this reduces to $\lambda_i\langle v_i, v_i\rangle = 0$. As $v_i \neq 0$, $\langle v_i, v_i\rangle \neq 0$, so $\lambda_i = 0$. ***(4 marks)***

(iii) **{U}**

$$\langle 2u+v, 2u-v\rangle = 4\langle u, u\rangle - 2\langle u, v\rangle + 2\langle v, u\rangle - \langle v, v\rangle = 4\|u\|^2 - \|v\|^2 = 0.$$

(3 marks)

(iv) (a) **{S}** Put $f_1 = 1$, $f_2 = x$, $f_3 = x^2$. Put $g_1 = f_1$, and try $g_2 = f_2 - \lambda f_1$, choosing λ so that g_2 is orthogonal to g_1 (i.e., to f_1). But $\langle 1, g_2\rangle = \int_{-1}^{1} x - \lambda\, dx = -2\lambda$, so this vanishes if $\lambda = 0$. Thus $g_2 = x$. Now put $g_3 = f_3 - \lambda f_1 - \mu f_2 = x^2 - \mu x - \lambda$ and choose the constants so that g_3 is orthogonal to f_1 and to f_2.

$$\langle g_3, f_1\rangle = \int_{-1}^{1} x^2 - \mu x - \lambda\, dx = 2/3 - 2\lambda,$$

so $\lambda = 1/3$, and

$$\langle g_3, f_2\rangle = \int_{-1}^{1} x^3 - \mu x^2 - \lambda x\, dx = -2\mu/3,$$

so $\mu = 0$. Thus $g_3 = x^2 - 1/3$. ***(7 marks)***

(b) **{S}** The subspace of linear polynomials is spanned by $\{1, x\}$, and this is a strictly orthogonal set. Then we simply compute $\pi(f)$, where $f(x) = 1 - x + x^2$, as

$$\frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\langle f, x\rangle}{\langle x, x\rangle}x,$$

and we have: $\langle f, 1\rangle = \int_{-1}^{1} 1 - x + x^2\, dx = 8/3$, $\langle f, x\rangle = \int_{-1}^{1} x - x^2 + x^3\, dx = -2/3$, $\langle 1, 1\rangle = \int_{-1}^{1} 1\, dx = 2$, $\langle x, x\rangle = \int_{-1}^{1} x^2\, dx = 2/3$, so the nearest vector is $-x + \frac{4}{3}$. ***(5 marks)***

5 (i) **{B/S}** For $v, w \in V$, $|\langle v, w\rangle| \leq \|v\|\,\|w\|$, with equality if and only if v and w are linearly dependent.

Put $g(x) = 3x - 1$. Then

$$\|g\|^2 = \langle g, g\rangle = \int_0^1 (3x-1)^2 dx = \int_0^1 9x^2 - 6x + 1\,dx = 1.$$

Also, $\|f\|^2 = \displaystyle\int_0^1 f(x)^2\,dx$, and $\langle f, g\rangle = \displaystyle\int_0^1 (3x-1)f(x)\,dx$, and the result follows from the Cauchy-Schwarz inequality. We have equality if $f(x) = a(3x-1)$, and in this case $\|f\|^2 = a^2\langle 3x-1, 3x-1\rangle = a^2$, so we take $a = 2$, and $f(x) = 6x - 2$. ***(9 marks)***

(ii) (a) **{B}** It is orthogonal (standard Fourier theory), but not orthonormal since, for example, $\langle 1, 1\rangle = 2\pi$. ***(2 marks)***

(b) **{S}** We have $\sin t \cos 5t = (\sin 6t - \sin 4t)/2$ and $\sin 3t \cos t = (\sin 4t + \sin 2t)/2$. Then

$$\langle \sin t \cos 5t, \sin 3t \cos t\rangle = -\langle sin 4t, \sin 4t\rangle/4 = -\pi/4,$$

and

$$\langle \sin t \cos 5t, \sin t \cos 5t\rangle = \langle \sin 3t \cos t, \sin 3t \cos t\rangle = \pi/2,$$

so that the cosine of the angle between the two functions is $\dfrac{-\pi/4}{\sqrt{\pi/2.\pi/2}} = -1/2$, so the angle is $2\pi/3$. ***(8 marks)***

(c) **{S}** We have $\langle \phi(v), A\rangle = \alpha a + \beta b - \beta c + \gamma d$, so we want to define $\hat{\phi}(A) = p + q\cos t + r \sin t$ so that $\langle \hat{\phi}(A), v\rangle$ agrees with the value above. But

$$\langle p + q\cos t + r\sin t, \alpha + \beta \cos t + \gamma \sin t\rangle = 2\pi p\alpha + \pi q\beta + \pi r\gamma,$$

so we need $p = \frac{a}{2\pi}$, $q = \frac{b-c}{\pi}$, $r = \frac{d}{\pi}$. Thus

$$\hat{\phi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a + 2(b-c)\cos t + 2d \sin t}{2\pi}.$$ ***(6 marks)***

End of Question Paper