



Answer **four** questions. If you answer more than four questions, only your best four will be counted.

- 1 (i) Let V be a vector space, and let \mathcal{V} denote the collection of vectors v_1, \dots, v_n in V . Define what it means for \mathcal{V} to
- (a) be linearly independent;
 - (b) span V ;
 - (c) be a basis for V . (6 marks)
- (ii) For each of the following subsets of \mathbb{R}^3 , say whether it is a subspace, justifying your answer.
- (a) $U_1 = \{(1, 2, 3)^T\}$;
 - (b) $U_2 = \{(x, y, z)^T \in \mathbb{R}^3 \mid y = 3\}$;
 - (c) $U_3 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x \leq z\}$;
 - (d) $U_4 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x + 3y + z = 0\}$. (7 marks)
- (iii) If U and W are subspaces of a vector space V , define $U + W$ and show that it is a subspace of V . (7 marks)
- (iv) In the vector space $V = \mathbb{R}^3$, let U be the subspace $\{(a, 2a, a)^T \mid a \in \mathbb{R}\}$, and W be the subspace $\{(x, y, z)^T \mid x = y\}$. Prove that $U \cap W = \{0_V\}$, and $V = U + W$. (5 marks)

- 2** (i) State the Steinitz Exchange Lemma. *(2 marks)*
- (ii) Let $\mathcal{V} = \{v_1, \dots, v_m\}$ be a set of vectors in an n -dimensional vector space V . Say whether or not each of the following statements is necessarily true, and, if not, give a counterexample:
- (a) If \mathcal{V} is linearly independent, then $m \leq n$;
 - (b) If $m \leq n$, then \mathcal{V} is linearly independent;
 - (c) If \mathcal{V} is a basis, then $m = n$;
 - (d) If $m = n$, then \mathcal{V} is a basis. *(6 marks)*

(iii) Let V and W be vector spaces and let $\phi : V \rightarrow W$ be a linear map. Define the kernel of ϕ and prove that ϕ is injective if and only if $\ker(\phi) = \{0_V\}$. *(7 marks)*

(iv) Let V be a finite dimensional vector space and let U and W be subspaces of V . Give an equation relating the dimensions of $U \cap W$, U , W and $U + W$. *(1 mark)*

(v) For $V = M_3(\mathbb{R})$ and $\mathbf{v} = (1, 0, 0)^T$, let

$$U = \{A \in V \mid A^T = -A\},$$

$$W = \{A \in V \mid A\mathbf{v} = 0\}.$$

You may assume that U and W are subspaces of V .

- (a) Find the dimensions of U , W , $U \cap W$ and $U + W$. *(8 marks)*
- (b) Write down an element of V which is not in $U + W$. *(1 mark)*

3 Consider the vector space $V = \mathbb{R}[x]_{\leq 3}$ of real polynomials in the variable x of degree at most 3.

- (i) Explain what the standard basis of the vector space V is and state the dimension of V . *(3 marks)*
- (ii) Show that the list of polynomials

$$1, x - 1, x^2 - x, x^3 - x^2$$

is another basis for V . *(4 marks)*

Consider the function $\phi : V \rightarrow V$ given by $\phi(f) = (1 + x)f'(x)$.

- (iii) Show that ϕ is a linear map. *(3 marks)*
- (iv) Find the matrix of ϕ with respect to the standard basis. *(3 marks)*
- (v) Find the matrix of ϕ with respect to the basis in part (ii). *(5 marks)*
- (vi) What is the kernel of ϕ ? *(2 marks)*
- (vii) Show that the image of ϕ is $\{f \in V \mid f(-1) = 0\}$. *(5 marks)*

- 4 (i) Let $V = \mathbb{R}[x]$, the vector space of real polynomials in the variable x , with the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

Let U be the subspace of V of spanned by x, x^2, x^3 .

- (a) Apply the Gram-Schmidt process to produce an orthogonal basis for U . **(6 marks)**
- (b) Calculate the closest element of U to the constant polynomial 1. **(6 marks)**
- (c) What is the distance from this element to 1? **(5 marks)**
- (ii) Let V be an inner product space (over \mathbb{R}).
- (a) State the Cauchy-Schwarz inequality for vectors $v, w \in V$, including the criterion for when equality holds. **(3 marks)**
- (b) By considering $C[0, \pi]$ with the inner product

$$\langle f, g \rangle = \int_0^\pi f(t)g(t) dt,$$

show that

$$\int_0^\pi \sqrt{\sin t} dt \leq \sqrt{2\pi}.$$

(5 marks)

- 5 (i) Explain what the standard inner product on $M_2(\mathbb{R})$ is and calculate the distance from $A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$ to $B = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$. **(5 marks)**
- (ii) Consider the Fourier inner product space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \rightarrow \mathbb{R}$ with the inner product

$$\langle f, g \rangle = \int_{-\pi}^\pi f(t)g(t) dt.$$

- (a) Is the set $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \dots\}$ orthogonal? Is it orthonormal? Justify your answers briefly. **(2 marks)**
- (b) Calculate the angle between $\cos 2t \cos t$ and $\cos 4t \cos t$. **(8 marks)**
- (iii) Let V be the subspace of $C[-\pi, \pi]$ spanned by the set $\mathcal{V} = \{1, \cos 3t, \sin 3t\}$. Consider the linear map $D : V \rightarrow V$ given by differentiation, $D(f) = f'$.
- (a) What is the matrix of D with respect to the basis \mathcal{V} ? **(3 marks)**
- (b) If \widehat{D} denotes the adjoint of $D : V \rightarrow V$, show that $\widehat{D} = -D$. **(7 marks)**

End of Question Paper