MAS277 Vector Spaces and Fourier Theory: Appendix

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January 17, 2011

1 Appendix: the L^2-convergence theorem

In this appendix, we will give a complete proof of Theorem 3.39: for any $f \in C^{2\pi}$, we have $\|\epsilon_n(f)\| \to 0$ as $n \to \infty$.

We begin with the Weierstrass Approximation Theorem.

Theorem 1.1 (Weierstrass) *Let $f \in C[0,1]$, and choose any $\epsilon > 0$. Then there is a polynomial p such that*

$$|f(x) - p(x)| < \epsilon$$

for all $x \in [0,1]$.

Proof. Define the polynomial $B_n(f)$ by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

Observe that it has degree n. It is a relatively simple check that if $f(x) = ax + b$, then $B_n(f) = f$ for all n, and if $f(x) = x^2$, then $B_n(f)(x) = f(x) + \frac{1}{n}x(1-x)$. Combining these, we see that

$$\sum_{k=0}^{n}\left(x - \frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{1}{n}x(1-x) \le \frac{1}{4n}$$

for all $x \in [0,1]$.

Now, given $\delta > 0$, and $x \in [0,1]$, consider the set F_x of integers k in $\{0,\dots,n\}$ with $|x - k/n| \ge \delta$. Then for such k, $1 \le (x-k/n)^2/\delta^2$, and so

$$\begin{aligned}
\sum_{k \in F_x}\binom{n}{k}x^k(1-x)^{n-k} &\le \frac{1}{\delta^2}\sum_{k\in F_x}\left(x-\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k} \\
&\le \frac{1}{\delta^2}\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k} \\
&\le \frac{1}{4n\delta^2}.
\end{aligned}$$

Now choose $\epsilon > 0$. There is some $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/2$ if $|x - y| < \delta$. As $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1$ by the binomial theorem, we have

$$\begin{aligned} & \left| f(x) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \right| \\ = & \left| \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right) \right) \binom{n}{k} x^k (1-x)^{n-k} \right| \\ \leq & \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

Fix n, and $x \in [0,1]$, and let F_x again denote the set of integers k in $\{0, \ldots, n\}$ with $|x - k/n| \geq \delta$. Then $|f(x) - f(k/n)| < \epsilon/2$ for $k \notin F_x$, while $|f(x) - f(k/n)| < M$ for $k \in F_x$ (where M is some bound). Then

$$\begin{aligned} |f(x) - B_n(f)(x)| & \leq \frac{\epsilon}{2} \sum_{k \notin F_x} \binom{n}{k} x^k (1-x)^{n-k} + M \sum_{k \in F_x} \binom{n}{k} x^k (1-x)^{n-k} \\ & < \frac{\epsilon}{2}.1 + \frac{M}{4n\delta^2} \\ & < \epsilon \end{aligned}$$

if $n > M/2\epsilon\delta^2$. For such n, we can take $p = B_n$, and we have shown the result. □

We can use this to deduce a similar result for trigonometric polynomials. Recall that the even trigonometric polynomials are those spanned by $\{\cos nt\}$, or equivalently by $\{\cos^n t\}$.

Lemma 1.2 *Let $f \in C^{2\pi}$ be an even function. Then for any $\epsilon > 0$, there is an even trigonometric polynomial T such that $|f(t) - T(t)| < \epsilon$ for all $t \in [-\pi, \pi]$.*

Proof. This follows easily from the Weierstrass Approximation Theorem 1.1: Note that $g(y) = f(\cos^{-1}(2y - 1))$ defines a continuous function for $y \in [0,1]$. Then Theorem 1.1 shows that there exists a polynomial $p(y)$ with $|f(\cos^{-1}(2y-1)) - p(y)| < \epsilon$ for all $y \in [0,1]$. Change the variable back again, so that $y = (1 + \cos t)/2$; write $T(t) = p((1 + \cos t)/2)$, which is a polynomial in $\cos t$, and therefore may be written as an even trigonometric polynomial, and $|f(t) - T(t)| < \epsilon$ for all $t \in [-\pi, \pi]$. □

Theorem 1.3 *Let $f \in C^{2\pi}$. Then for any $\epsilon > 0$, there is a trigonometric polynomial T such that $|f(t) - T(t)| < \epsilon$ for all $t \in [-\pi, \pi]$.*

Proof. Note that Lemma 1.2 proves what we want in the special case where the function is even. We will deduce the full result by applying the special case several times. Let $f \in C^{2\pi}$ and fix $\epsilon > 0$. Note that $f(t) + f(-t)$ and $[f(t) - f(-t)] \sin t$ are both even functions. By Lemma 1.2 there are trigonometric polynomials $T_1(t)$ and $T_2(t)$ such that

$$\begin{aligned} |(f(t) + f(-t)) - T_1(t)| & < \epsilon/2 \\ |([f(t) - f(-t)] \sin t) - T_2(t)| & < \epsilon/2 \end{aligned}$$

for all $t \in [-\pi, \pi]$. Let $f(t)+f(-t) = T_1(t)+d_1(t)$, and $[f(t)-f(-t)]\sin t = T_2(t)+d_2(t)$ so that $|d_1(t)| < \frac{\epsilon}{2}$ and $|d_2(t)| < \frac{\epsilon}{2}$. Multiply the first equation by $\sin^2 t$, the second by $\sin t$, add and divide by 2 to get:

$$f(t)\sin^2 t = T_3(t) + d_3(t),$$

where $T_3(t) = [T_1(t)\sin^2 t + T_2(t)\sin t]/2$ and $d_3(t) = [d_1(t)\sin^2 t + d_2(t)\sin t]/2$. Then

$$\begin{aligned} |d_3(t)| &= \left|\frac{d_1(t)\sin^2 t + d_2(t)\sin t}{2}\right| \\ &\leq \left|\frac{d_1(t)\sin^2 t}{2}\right| + \left|\frac{d_2(t)\sin t}{2}\right| \\ &\leq \left|\frac{d_1(t)}{2}\right| + \left|\frac{d_2(t)}{2}\right| \\ &< \epsilon/2, \end{aligned}$$

the second inequality holding as $|\sin t| \leq 1$ for all t. Thus

$$f(t)\sin^2 t = T_3(t) + d_3(t), \tag{1.1}$$

where $|d_3(t)| < \frac{\epsilon}{2}$ for all $t \in [-\pi, \pi]$. But this result is true for any $f \in C^{2\pi}$. So as well as being true for $f(t)$, it is also true for $g(t) = f(t - \frac{\pi}{2})$. Thus $f(t-\frac{\pi}{2})\sin^2 t = T_4(t) + d_4(t)$, where $|d_4(t)| < \frac{\epsilon}{2}$. Change variable $t \mapsto t + \frac{\pi}{2}$ to deduce that

$$f(t)\cos^2 t = T_5(t) + d_5(t), \tag{1.2}$$

where $|d_5(t)| < \frac{\epsilon}{2}$ for all $t \in [-\pi, \pi]$.

Now we add the two results (1.1) and (1.2) to deduce that

$$f(t) = T(t) + d(t),$$

where $T(t) = T_3(t) + T_5(t)$ is a trigonometric polynomial, and $d(t) = d_3(t) + d_5(t)$ satisfies

$$|d(t)| = |d_3(t) + d_5(t)| \leq |d_3(t)| + |d_5(t)| < \epsilon$$

for all $t \in C[-\pi, \pi]$, as required.

Recall that $\pi_n(f)$ is the closest point to f lying in $\mathcal{T}_{\leq n}$, so $\|f - \pi_n(f)\|$ can be regarded as the distance from f to $\mathcal{T}_{\leq n}$. The L^2-convergence theorem says that by taking n to be sufficiently large, we can make this distance as small as we like. Let's explain that this is a corollary of the last theorem.

Indeed, let $\epsilon > 0$. By Theorem 1.3, there is a trigonometric polynomial p of some degree m such that $|f(t) - p(t)| < \epsilon$ for all $t \in [-\pi, \pi]$. However, for all $n \geq m$, $\pi_n(f)$ is the closest point of $\mathcal{T}_{\leq n}$ to f, so that $\|\epsilon_n(f)\| = \|f - \pi_n(f)\| \leq \|f - p\|$. But

$$\|f - p\|^2 = \int_{-\pi}^{\pi} (f(t) - p(t))^2\, dt < 2\pi\epsilon^2.$$

Therefore $\|\epsilon_n(f)\| < \epsilon\sqrt{2\pi}$ for such n. Since $\epsilon > 0$ was arbitrary, we conclude that $\|\epsilon_n(f)\| \to 0$ as $n \to \infty$, as required. □