MAS277: Vector Spaces and Fourier Theory

Semester 2, 2010/11; 10 credits

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The first part of the course introduces abstract vector spaces and linear transformations, building on the concrete examples covered in MAS201. Many results that were merely stated in MAS201, or proved by matrix methods, will be given conceptual proofs in this course. The abstract approach will allow us to give efficient proofs that simultaneously tell us interesting things about vectors, matrices, polynomials, sequences, differential equations, and many other objects. A central aim of the course is to help students become comfortable with the required level of abstraction. Next, we introduce inner product spaces, including the space of continuous periodic functions. This allows us to define distances and angles between functions, by analogy with distances and angles between vectors in $\mathbf{R}^3$. We then reinterpret the theory of Fourier series in these terms. We also discuss adjoints of operators. We show that any self-adjoint operator admits an orthonormal basis of eigenvectors, and that the eigenvalues are all real numbers.

Outline syllabus.

- Vector spaces, linear maps, subspaces.
- Independence and spanning sets.
- Linear maps out of $\mathbf{R}^n$; matrices for linear maps.
- Theorems about bases.
- Eigenvalues and eigenvectors.
- Inner products and the Cauchy-Schwartz inequality.
- Projections and the Gram-Schmidt procedure.
- Adjoints, and diagonalisation of self-adjoint operators.
- Fourier theory and the L^2 convergence theorem.

Aims.

- Introduce the abstract theory of vector spaces and linear maps between them.
- Introduce the abstract theory of inner product spaces.
- Reinterpret Fourier theory in terms of inner product spaces.
- Familiarise students with abstract and axiomatic mathematics.

Learning outcomes.

- Students should understand the main concepts of linear algebra: vector spaces; subspaces, (direct) sum and intersection; linear maps, kernels and images; independent sets, spanning sets and bases; relations with matrices.
- Students should be familiar with a range of examples of these concepts.
- Students should be able to prove the main theorems about these concepts.
- Students should understand the main concepts of inner product spaces: the Cauchy-Schwartz inequality, orthonormal sequences, orthogonal complements, projectors, Parseval's inequality, the Gram-Schmidt procedure.
- Students should be familiar with a range of examples of these concepts, including the space of continuous periodic functions.
- Students should understand how this relates to Fourier theory, including the statement of the L^2 convergence theorem.

Teaching methods. Lectures, tutorials, problem solving
There will be tutorials in week 1 for this module; important revision of material from MAS201 will be done.

Tutorial questions will be announced ahead of the tutorials (with the exception of week 1). You should attempt the questions before the tutorial.
I expect there will be 5 pieces of marked homework for this module, to be handed in in lectures in weeks 2, 4, 6, 8, 10 and returned at the tutorials in weeks 3, 5, 7, 9, 11.
Further questions will be set for homework and some solutions will be presented in lectures.
22 lectures, 6 tutorials, 0 practicals.

Assessment. One formal 2 hour written examination. Format: 4 questions from 5.

Full syllabus.

1. Vector spaces. Definitions and examples.

2. Linear maps. Definitions and examples.

3. Subspaces. Definitions and examples, (direct) sums and intersections.

4. Independence and spanning sets. Definitions and examples. Bases.

5. Linear maps out of $\mathbf{R}^n$. A linear map $\mathbf{R}^n \to V$ is the same as a list of n elements of V.

6. Matrices for linear maps. Definitions, properties, and behaviour under change of basis.

7. Theorems about bases. Invariance of dimension, rank-nullity formula, $\dim(U+V)+\dim(U\cap V) = \dim(U)+\dim(V)$.

8. Eigenvalues and eigenvectors. A brief account for abstract vector spaces.

9. Inner products. Definitions and examples.

10. The Cauchy-Schwartz inequality.

11. Projections and the Gram-Schmidt procedure.

12. Hermitian forms.

13. Adjoints of linear maps. Definition, proof of existence and uniqueness.

14. Diagonalisation of self-adjoint operators. Self-adjoint operators have real eigenvalues, and admit an orthonormal basis of eigenvectors.

15. Fourier theory. Finite Fourier series are orthogonal projections onto spaces of trigonometric polynomials. Parseval's inequality. Statement of the L^2 convergence theorem (with proof outlined in a non-examinable appendix).

Recommended books.

C P.Halmos "Finite-dimensional vector spaces" (Shelfmark 512.83)
C W. Keith Nicholson "Linear algebra with applications" (Shelfmark 512.5)