MAS277 Vector Spaces and Fourier Theory

Hand-out: Reminders on trigonometric identities and their use to calculate integrals in Fourier Theory

Trigonometric identities

I hope that you all remember the following two trigonometric identities from school:

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ \cos(A+B) &= \cos A\cos B-\sin A\sin B.\end{aligned}$$

These are the only two identities you ever need to learn!

(And if you can't remember those, you can derive them from $e^{i(A+B)}=e^{iA}e^{iB}$, together with $e^{i\theta}=\cos\theta+i\sin\theta$.)

Other trig identities can be derived from these. In particular, here's how you get various double angle formulae. These are useful for integrals which are important in Fourier Theory.

First consider the two identities above, and replace B by $-B$. Then we get

$$\begin{aligned}\sin(A-B) &= \sin A\cos B-\cos A\sin B\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B.\end{aligned}$$

These follow immediately, since $\cos(-B)=\cos B$ and $\sin(-B)=-\sin B$. We can then combine the two relations:

$$\begin{aligned}\cos(A+B) &= \cos A\cos B-\sin A\sin B\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B,\end{aligned}$$

to get the very useful identities:

$$\cos A\cos B=(\cos(A+B)+\cos(A-B))/2$$

and

$$\sin A\sin B=(\cos(A-B)-\cos(A+B))/2,$$

which follow simply by adding and subtracting these two identities. Similarly, the two other identities

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ \sin(A-B) &= \sin A\cos B-\cos A\sin B\end{aligned}$$

imply the identity

$$\sin A\cos B=(\sin(A+B)+\sin(A-B))/2.$$

You could learn the identities:

$$\begin{aligned}\cos A\cos B &= (\cos(A+B)+\cos(A-B))/2\\ \sin A\sin B &= (\cos(A-B)-\cos(A+B))/2\\ \sin A\cos B &= (\sin(A+B)+\sin(A-B))/2,\end{aligned}$$

but I think that they are easy to mix up. So I find it more reliable to remember how to derive them.

Use in Fourier Theory

Often you may be asked in Fourier theory to take the inner product of, say, $\cos mt$ and $\cos nt$ in the Fourier inner product space $C[-\pi, \pi]$ (with m and n integers). Well, we have

$$\langle \cos mt, \cos nt \rangle = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt.$$

Until we remember our identity, this probably looks quite tricky. But with it, it becomes almost trivial:

$$\begin{aligned} \langle \cos mt, \cos nt \rangle = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt &= \int_{-\pi}^{\pi} \frac{1}{2} \left(\cos(m+n)t + \cos(m-n)t \right) dt \\ &= \left[\frac{\sin(m+n)t}{2(m+n)} + \frac{\sin(m-n)t}{2(m-n)} \right]_{-\pi}^{\pi} \end{aligned}$$

If we recall that $\sin k\pi = 0$ for any integer k, we see that $\sin(m+n)\pi = \sin(m-n)\pi = 0$. As long as we aren't dividing by 0, we see that $\langle \cos mt, \cos nt \rangle = 0$.

But we have to be careful: if $m = n$, then the second term in the sum is $\frac{0}{0}$ (and if $m = n = 0$, then the first term is also of that form).

Luckily, we can do the integral separately in these cases. So suppose $m = n$. Then

$$\begin{aligned} \langle \cos mt, \cos mt \rangle = \int_{-\pi}^{\pi} \cos^2 mt \, dt &= \int_{-\pi}^{\pi} \frac{1}{2} \left(\cos 2mt + 1 \right) dt \\ &= \left[\frac{\sin 2mt}{4m} + \frac{t}{2} \right]_{-\pi}^{\pi}, \end{aligned}$$

and if $m > 0$, the first term vanishes as before, giving $\langle \cos mt, \cos mt \rangle = \pi$. The case $m = 0$ is easy:

$$\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \, dt = 2\pi.$$

The same methods allow you to compute all inner products used in Fourier theory. Since you (almost certainly) did this in MAS202, I'll just summarise the results:

$$\begin{aligned} \langle \cos mt, \cos nt \rangle &= \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n > 0 \\ 2\pi, & \text{if } m = n = 0 \end{cases}, \\ \langle \sin mt, \sin nt \rangle &= \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n > 0 \\ 0, & \text{if } m = n = 0 \end{cases}, \\ \langle \sin mt, \cos nt \rangle &= 0, \quad \text{for all } m, n. \end{aligned}$$

Since these were all treated in MAS202, you may quote these without proof or any further explanation in the exam.