MAS221 Analysis, Semester 2 Solutions V
Solutions for Chapters 7, 8 and 9

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Chapter 7 Problems: Riemann Integration

113. (a)

$$\int_0^3 r(t)\,dt = 1 + e + e^4,$$

$$\int_0^3 s(t)\,dt = e + e^4 + e^9.$$

(b) The function f is continuous, and therefore Riemann integrable on a bounded interval.

(c) We have that the function r (from part (a)) is a step function with $r \leq f$, and s is a step function with $s \geq f$.
Hence

$$\int_0^3 f(t)\,dt = L\int_0^3 f(t)\,dt \geq \int_0^3 r(t)\,dt = 1 + e + e^4$$

and

$$\int_0^3 f(t)\,dt = U\int_0^3 f(t)\,dt \leq \int_0^3 s(t)\,dt = e + e^4 + e^9.$$

114. (a) Let $k/n \leq x < (k+1)/n$. Then $(k/n)^2 \leq x^2$, from which it follows that $r_n(t) \leq f(t)$ for $t \in [0,1]$.
Now, by definition

$$\int_0^1 r_n(t)\,dt = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2),$$

so, by using the given formula

$$\int_0^1 r_n(t)\,dt = \frac{1}{6n^3} n(n-1)(2n-1) = \frac{1}{6n^2}(n-1)(2n-1).$$

(b) Similarly to the above, $s_n(t) \geq f(t)$ for all $t \in [0,1]$. By definition

$$\int_0^1 s_n(t)\,dt = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2),$$

so, by using the given formula

$$\int_0^1 r_n(t)\,dt = \frac{1}{6n^3} n(n+1)(2n+1) = \frac{1}{6n^2}(n+1)(2n+1).$$

(c) We have, for each n,

$$\int_0^1 r_n(t)\,dt \leq L\int_0^1 f(t)\,dt \leq U\int_0^1 f(t)\,dt \leq \int_0^1 s_n(t)\,dt.$$

Observe that

$$\lim_{n\to\infty} \int_0^1 r_n(t)\,dt = \lim_{n\to\infty} \frac{1}{6}(1-1/n)(2-1/n) = \frac{1}{3}$$

and

$$\lim_{n\to\infty} \int_0^1 s_n(t)\,dt = \lim_{n\to\infty} \frac{1}{6}(1+1/n)(2+1/n) = \frac{1}{3}.$$

It follows by the sandwich rule that

$$L\int_0^1 f(t)\,dt = U\int_0^1 f(t)\,dt = \int_0^1 f(t)\,dt = \frac{1}{3}.$$

115. (a) The function f is continuous on the bounded interval $[1,2]$ so it is Riemann integrable.

(b) Note that the function k is *not* a step function, as a step function is defined to be a sum of *finitely many* constant multiples of characteristic functions. The function k here is a sum of infinitely many such.

Nonetheless k is Riemann integrable, as we can approximate from above and from below by step functions, with integrals which converge to the same thing. For example, we could take, for $n \geq 1$,

$$r_n = \sum_{i=2}^{n} \frac{2^i - 1}{2^i} \chi_{[\frac{2^i-1}{2^i}, \frac{2^{i+1}-1}{2^{i+1}})}$$

and

$$s_n = r_n + \chi_{[\frac{2^{n+1}-1}{2^{n+1}}]}.$$

Then r_n and s_n are step functions with $r_n(x) \leq k(x) \leq s_n(x)$. Thus for all n,

$$\int_0^1 r_n(x)\,dx \leq L\int_0^1 k(x)\,dx \leq U\int_0^1 k(x)\,dx \leq \int_0^1 s_n(x)\,dx$$
$$= \int_0^1 r_n(x)\,dx + \frac{1}{2^{n+1}}.$$

Since $\lim_{n\to\infty} \frac{1}{2^{n+1}} = 0$, by the sandwich rule, we have $L\int_0^1 k(x)\,dx = U\int_0^1 k(x)\,dx$ and k is Riemann integrable.

116. (a) Let

$$r(t) = \sum_{i=1}^{m} \alpha_i \chi_{I_i}(t), \qquad s(t) = \sum_{j=1}^{n} \beta \chi_{J_j}(t).$$

Then

$$(rs)(t) = r(t)s(t) = \sum_{i,j=1}^{m,n} \alpha_i \beta_j \chi_{I_i}(t)\chi_{J_j}(t).$$

Now

$$\chi_{I_i}(t)\chi_{J_j}(T) = \chi_{I_i \cap J_j}(t)$$

and the intersection of two intervals is again an interval. Thus rs is a step function.

(b) Let $\alpha, \beta \in \mathbb{R}$. Then $(\alpha r(t) + \beta s(t))^2 \geq 0$ for all t, so

$$\int_a^b (\alpha r(t) - \beta s(t))^2\,dt \geq 0$$

that is

$$\alpha^2 \int_a^b r(t)^2\,dt - 2\alpha\beta \int_a^b r(t)s(t)\,dt + \beta^2 \int_a^b s(t)^2\,dt \geq 0$$

and

$$2\alpha\beta \int_a^b r(t)s(t)\,dt \leq \alpha^2 \int_a^b r(t)^2\,dt + \beta^2 \int_a^b s(t)^2\,dt.$$

Let

$$\alpha = \left(\int_a^b s(t)^2\,dt\right)^{\frac{1}{2}}, \qquad \beta = \left(\int_a^b r(t)^2\,dt\right)^{\frac{1}{2}}.$$

Then we see

$$\left(\int_a^b s(t)^2\,dt\right)^{\frac{1}{2}} \left(\int_a^b r(t)^2\,dt\right)^{\frac{1}{2}} \int_a^b r(t)s(t)\,dt \leq \int_a^b r(t)^2\,dt \int_a^b s(t)^2\,dt.$$

Rearranging and squaring, the result follows.

(c) By the above

$$\begin{aligned}\|f+g\|^2 &= \int_a^b f(t)^2 + 2f(t)g(t) + g(t)^2\,dt \\ &\le \int_a^b f(t)^2\,dt + 2\left(\int_a^b f(t)^2\,dt\right)^{\frac{1}{2}}\left(\int_a^b g(t)^2\,dt\right)^{\frac{1}{2}} + \int_a^b g(t)^2\,dt,\end{aligned}$$

that is to say,

$$\|f+g\|^2 \le \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2.$$

Thus

$$\|f+g\| \le \|f\| + \|g\|.$$

117. We have

$$\int_{a(x)}^{b(x)} f(y)\ dy = \int_0^{b(x)} f(y)\ dy - \int_0^{a(x)} f(y)\ dy.$$

Write $F(x) = \int_0^x f(y)\,dy$. Then by the fundamental theorem of calculus, $F'(x) = f(x)$. Then

$$\int_0^{b(x)} f(y)\ dy = F(b(x)),$$

and using the chain rule

$$\frac{d}{dx}\int_0^{b(x)} f(y)\ dy = F'(b(x))b'(x) = f(b(x))b'(x).$$

Similarly,

$$\frac{d}{dx}\int_0^{a(x)} f(y)\ dy = f(a(x))a'(x).$$

Subtracting these

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(y)\ dy = f(b(x))b'(x) - f(a(x))a'(x),$$

as required.

118. (a) It follows immediately from the fundamental theorem of calculus that L is differentiable, with $L'(x) = 1/x$.

(b) We have

$$L(xy) = \int_1^{xy} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt.$$

In the second integral, set $t = xu$, so $dt = x\,du$. When $t = x$, $u = 1$, and when $t = xy$, $u = y$, so

$$L(xy) = \int_1^x \frac{1}{t}\,dt + \int_1^y \frac{1}{xu}x\ du = \int_1^x \frac{1}{t}\,dt + \int_1^y \frac{1}{u}\ du = L(x) + L(y).$$

(c) By the above $L(x/y) = L(x) + L(1/y)$. So it suffices to check that $L(1/y) = -L(y)$. We have

$$L(1/y) = \int_1^{1/y} \frac{1}{t}\, dt.$$

Set $t = u/y$, so $dt = 1/y\ du$. When $t = 1$, $u = y$. When $t = 1/y$, $u = 1$, so

$$L(1/y) = \int_y^1 \frac{y}{u}\frac{1}{y}\, du = -\int_1^y \frac{1}{u}\, du = -L(y),$$

as required.

119. (a) We have $f(xy) = f(x) + f(y)$ for all $x, y > 0$. We claim that $f(x^n) = nf(x)$ whenever $n \in \mathbb{N}$. We work by induction.

The result certainly holds when $n = 1$. Suppose the result holds when $n = k$. Then $f(x^k) = kf(x)$. So

$$f(x^{k+1}) = f(x^k \cdot x) = f(x^k) + f(x) = kf(x) + f(x) = (k+1)f(x).$$

So the result holds when $n = k + 1$. Thus it holds for all $n \in \mathbb{N}$ by induction.

Observe $f(1) = f(1 \times 1) = f(1) + f(1)$. Therefore $f(1) = 0$, and $f(x^0) = 0f(x)$ for any $x > 0$. Now

$$f(1/x) + f(x) = f(x/x) = f(1) = 0$$

so $f(1/x) = -f(x)$. Hence, for $n \in \mathbb{N}$.

$$f(x^{-n}) = f(1/x^n) = -f(x^n) = -nf(x).$$

It follows that $f(x^k) = kf(x)$ for all $k \in \mathbb{Z}$.

Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Then $(x^{a/b})^b = x^a$. By the above

$$bf(x^{a/b}) = f((x^{a/b})^b) = f(x^a) = af(x)$$

so $f(x^{a/b}) = \frac{a}{b}f(x)$, and $f(x^q) = qf(x)$ for all $q \in \mathbb{Q}$.

Let $a \in \mathbb{R}$. Then we have a sequence (a_n) in $\mathbb{Q}$ converging to a. By continuity of f and the above

$$f(x^a) = \lim_{n\to\infty} f(x^{a_n}) = \lim_{n\to\infty} a_n f(x) = af(x).$$

(b) Let $x > 0$. Observe

$$f(x) = f(e^{\ln x})) = f(e)\ln x$$

by the above.

(c) By part (b), it suffices to prove $L(e) = 1$. We have

$$L(e) = \int_1^e \frac{1}{t}\, dt.$$

Set $t = e^u$, so $dt = e^u\, du$. At $t = 1$, we have $u = 0$. At $t = e$, we have $u = 1$. So

$$L(e) = \int_0^1 \frac{1}{e^u} e^u\, du = \int_0^1 du = 1$$

as required.

120. Define $G\colon [a, b] \to \mathbb{R}$ by

$$G(x) = \int_a^x f(t)\, dt$$

Then G is a continuous function. This was mentioned in the lectures. To see why, let $h > 0$. Since the function f is Riemann integrable, it is bounded, so we have $M \in \mathbb{R}$ such that $|f(t)| \leq M$ for all $t \in [a, b]$. Then

$$|G(x+h) - G(x)| = \left| \int_x^{x+h} f(t)\, dt \right| \leq \int_x^{x+h} |f(t)|\, dt \leq Mh.$$

Similarly $|G(x-h) - G(x)| \leq Mh$.

So $G(x+h) \to G(x)$ as $h \to 0$, and G is continuous.

Thus we can define a continuous function $F\colon [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt - \int_x^b f(t)\, dt.$$

Observe that

$$F(a) = -\int_a^b f(t)\, dt, \qquad F(b) = \int_a^b f(t)\, dt = -F(a).$$

If $\int_a^b f(t)\, dt = 0$, then we can take $x = a$ and we have the required result. Otherwise, either $F(a) < 0$ and $F(b) > 0$ or $F(a) > 0$ and $F(b) < 0$.

Either way, by the intermediate value theorem, we have $x \in [a, b]$ such that $F(x) = 0$, that is to say

$$\int_a^x f(t)\, dt = \int_x^b f(t)\, dt.$$

121. (a) We have

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Let $\mu = \frac{1}{b-a}\int_a^b f(x)\,dx$. Then $\int_a^b f(x)\,dx = (b-a)\mu$ and $m \leq \mu \leq M$, so $\mu \in [m, M]$.

(b) Let $m = \inf\{f(x) \mid x \in [a,b]\}$ and $M = \sup\{f(x) \mid x \in [a,b]\}$. Then by part (a) we have $\mu \in [m, M]$ where

$$\int_a^b f(x)\,dx = (b-a)\mu.$$

But as f is a continuous function on a closed bounded interval, m is the minimum value of f, and M is the maximum value. As $\mu \in [m, M]$ and f is continuous, by the intermediate value theorem there is some $\xi \in [a,b]$ such that $f(\xi) = \mu$. Hence

$$\int_a^b f(x)\,dx = (b-a)f(\xi).$$

122. Let $m = \inf\{f(x)\ x \in [a,b]\}$ and $M = \sup\{f(x) \mid x \in [a,b]\}$. Then $m \leq f(x) \leq M$ for all $x \in [a,b]$. As $g(x) \geq 0$, we have

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

for all $x \in [a,b]$, and so

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

Let

$$I = \int_a^b g(x)\,dx.$$

If $I = 0$, then the above tells us that

$$\int_a^b f(x)g(x)\,dx = 0$$

and any $\xi \in [a,b]$ satisfies

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

As $g(x) \geq 0$, we need to prove the result when $I > 0$. Then

$$m \leq \frac{1}{I}\int_a^b f(x)g(x)\,dx \leq M.$$

By the intermediate value theorem, as in the solution to Q121(b), we have $\xi \in [m, M]$ such that

$$f(\xi) = \frac{1}{I} \int_a^b f(x)g(x)\, dx$$

or in other words

$$\int_a^b f(x)g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

We do need the assumption $g(x) \geq 0$. To see this, consider the function $g(x) = x$ for $x \in [-1, 1]$. Then

$$\int_{-1}^1 g(x)\, dx = 0,$$

so the result would say that for any continuous function $f\colon [-1, 1] \to \mathbb{R}$ we have

$$\int_{-1}^1 x f(x)\, dx = 0.$$

But this is clearly not the case, taking for example $f(x) = x$.

Chapter 8 Problems: Sequences and series of functions

126. For $x \in [0, \pi]$ we have $0 \leq \sin x < 1$ except when $x = \pi/2$, where $\sin x = 1$. Thus $f_n(x)$ converges pointwise to the function $f\colon [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \neq \pi/2, \\ 1 & x = \pi/2. \end{cases}$$

Each function f_n is continuous. The function f is not continuous. So the sequence (f_n) does not converge uniformly.

127. (a) Observe that (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

Each function (f_n) is continuous, but the function f is not continuous at 0, so the convergence cannot be uniform.

(b) Let $x \in \mathbb{R}$. Then for any $n \geq x$, we have $f_n(x) = 0$, so the sequence (f_n) converges pointwise to the zero function.
On the other hand

$$M_n = \sup\{|f_n(t) - 0| \mid t \in \mathbb{R}\} = \infty,$$

so certainly (M_n) does not converge to zero, and the sequence (f_n) therefore does not converge uniformly to the zero function.

(c) For $x > 1$, $1/x^n \to 0$ as $n \to \infty$. Hence for each $x \in (1, \infty)$, $e^x/x^n \to 0$ as $n \to \infty$, and the sequence (f_n) converges pointwise to the zero function.

But $e^x/x^n \to \infty$ as $x \to \infty$, so

$$M_n = \sup\{|f_n(t) - 0| \mid t \in (1, \infty)\} = \infty.$$

So certainly (M_n) does not converge to zero, and the sequence (f_n) therefore does not converge uniformly to the zero function.

(d) We know that $e^{-k} \to 0$ as $k \to \infty$, so $f_n(x) \to 0$ as $n \to \infty$ if $x \neq 0$. If $x = 0$, then $f_n(x) = 1$ for all n. So (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

Each function (f_n) is continuous, but the function f is not continuous at 0, so the convergence cannot be uniform.

(e) The sequence (f_n) clearly does not converge pointwise.

128. For a sequence of functions $g_n : D \to \mathbb{R}$, we know that (g_n) converges uniformly to a function g if and only if, when we set

$$M_n = \sup\{|g_n(t) - g(t)| \mid t \in D\}$$

we have $M_n \to 0$ as $n \to \infty$.

(a) Certainly, for each x, $g_n(x) \to 0$ as $n \to \infty$, so the sequence (g_n) converges pointwise to the zero function.

On the interval $[0, 1]$,

$$M_n = \sup\{|g_n(t) - 0| \mid t \in [0, 1]\} = \frac{1}{n}$$

Note that $M_n \to 0$ as $n \to \infty$, so on the interval $[0, 1]$, the sequence (g_n) converges uniformly to the zero function.

On the other hand, on $[0, \infty)$,

$$M_n = \sup\{|g_n(t) - 0| \mid t \in [0, \infty)\} = \infty,$$

so the sequence (g_n) does not converge uniformly on $[0, \infty)$.

(b) Observe that

$$g_n(x) = \frac{x^n}{1 + x^n} = \frac{1}{1 + 1/x^n} \to \begin{cases} 0 & x < 1, \\ \frac{1}{2} & x = 1, \\ 1 & x > 1. \end{cases}$$

Thus the pointwise limit function is not continuous on $[0, 1]$ or on $[0, \infty)$, so convergence is not uniform on either interval.

(c) If $0 \le x \le 1$, then $n/x^n \to \infty$ as $n \to \infty$, so $g_n(x) \to 0$ as $n \to \infty$. If $x > 1$, then $n/x^n \to 0$ as $n \to \infty$, so $g_n(x) \to 1$ as $n \to \infty$. So (g_n) converges pointwise to

$$g(x) = \begin{cases} 0 & x \le 1, \\ 1 & x > 1. \end{cases}$$

On the interval $[0, 1]$,

$$M_n = \sup\{|g_n(t) - 0| \mid t \in [0, 1]\} = \frac{1}{1+n} \to 0$$

as $n \to \infty$. So the sequence (g_n) converges uniformly to the zero function on $[0, 1]$.

On the other hand the above pointwise limit g is not continuous on $[0, \infty)$, whereas each function g_n is continuous on $[0, \infty)$, so convergence is not uniform on $[0, \infty)$.

129. (a) Observe that, for all $x \in [0, 1]$, $h_n(x) \to 1$ as $n \to \infty$. So (h_n) converges pointwise to the constant function $h : [0, 1] \to \mathbb{R}$, $h(x) = 1$ for all x.

Let

$$M_n = \sup\{|h_n(x) - 1| \mid x \in [0, 1]\} = 1 - (1 - 1/n)^2 = \frac{1}{n^2} - \frac{2}{n}.$$

So $M_n \to 0$ as $n \to \infty$. So (h_n) converges uniformly to the constant function h.

(b) We know that $x^n \to 0$ as $n \to \infty$ if $0 \le x < 1$, and $x^n \to 1$ as $n \to \infty$ if $x = 1$. Hence (h_n) converges pointwise to the function h, where

$$h(x) = \begin{cases} x & 0 \le x < 1, \\ 0 & x = 1. \end{cases}$$

Each function h_n is continuous, and this limit is not continuous, so convergence is not uniform.

(c) The sequence $(h_n(x))$ has pointwise limit $\frac{1}{1-x}$ if $0 \le x < 1$ (geometric series), but it does not converge if $x = 1$. Because it does not converge when $x = 1$, the sequence of functions (h_n) does not converge pointwise on $[0, 1]$ (and so it certainly does not converge uniformly on $[0, 1]$).

130. (a) For each t, we have

$$f_n(t) = \frac{n + \cos x}{2n + \sin^2 x} = \frac{1 + (\cos x)/n}{2 + (\sin^2 x)/n} \to \frac{1}{2}$$

as $n \to \infty$. So the sequence (f_n) converges pointwise to the constant function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \frac{1}{2}$ for all $x \in \mathbb{R}$.

(b) Note that, for all $x \in \mathbb{R}$,

$$\left| f_n(x) - \frac{1}{2} \right| = \left| \frac{2n + 2\cos x - 2n - \sin^2 x}{2(2n + \sin^2 x)} \right| = \left| \frac{2\cos x - \sin^2 x}{2(2n + \sin^2 x)} \right| \\ \leq \frac{2+1}{2(2n)} = \frac{3}{4n} \to 0$$

as $n \to \infty$.

It follows that (f_n) converges to f uniformly.

(c) By the uniform convergence theorem for integrals

$$\lim_{n\to\infty} \int_2^7 f_n(x)\ dx = \int_2^7 \frac{1}{2}\ dx = \frac{5}{2}.$$

131. (a) As indicated in the question, we need to check continuity at $x = 0$ and $x = 1$.

Let $0 < x < 1$, so

$$f_n(x) = \frac{x^{1/n} - 1}{\ln x}.$$

We have $x^{1/n} - 1 \to -1 \neq 0$ as $x \to 0$, and $\ln x \to -\infty$ as $x \to 0$. So clearly

$$\frac{x^{1/n} - 1}{\ln x} \to 0 = f_n(0)$$

as $x \to 0$, and f_n is continuous at 0.

On the other hand,

$$x^{1/n} - 1 \to 1 - 1 = 0 \qquad \text{and} \qquad \ln x \to 0$$

as $x \to 1$, so by L'Hôpital's rule,

$$\lim_{x\to 1} \frac{x^{1/n} - 1}{\ln x} = \lim_{x\to 1} \frac{(1/n)x^{1/n-1}}{1/x} = \frac{1}{n} = f_n(1),$$

so f_n is continuous at 1.

(b) Since we can assume the function f_n is increasing, we have

$$|f_n(x)| \leq \frac{1}{n}$$

for all $x \in [0, 1]$. But $1/n \to 0$ as $n \to \infty$, so it follows that (f_n) converges uniformly to the zero function.

(c) The result now follows immediately by the uniform convergence theorem for integrals.

132. For $0 \le t \le 1$, we have
$$\left|\frac{e^{t^4}}{n}\right| \le \frac{1}{n} \to 0$$
as $n \to \infty$.

Hence the sequence of functions (e^{t^4}/n) converges uniformly to the zero function on $[0,1]$. By the uniform convergence theorem for integrals, it follows that
$$\lim_{n\to\infty} \int_0^1 \frac{e^{t^4}}{n}\, dt = 0.$$

For $1 \le t \le 2$, we have
$$|t^{2-(\sin nt)/n} - t^2| = |t^2||t^{-(\sin nt)/n} - 1| \le 4(t^{1/n} - 1) \le 4(2^{1/n} - 1) \to 0$$
as $n \to \infty$.

Hence the sequence of functions $(t^{2-(\sin nt)/n})$ converges uniformly on the interval $[1,2]$ to the function $f : [1,2] \to \mathbb{R}$ given by $f(t) = t^2$. It follows that
$$\lim_{n\to\infty} \int_1^2 t^{2-(\sin nt)/n}\, dt = \int_1^2 t^2\, dt = \left[\frac{t^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

133. (a) Let $0 < x < y \le \theta$. Then
$$\left|\frac{1}{x} - \frac{1}{y}\right| \ge \frac{|x-y|}{\theta^2}.$$
So if $|x-y| = \delta$, then we can choose $\theta = 1/\sqrt{\delta}$ so if $x, y \le \theta$ then $|1/x - 1/y| \ge 1$.
It follows that f is not uniformly continuous.

(b) The function g is continuous, with a closed bounded domain, and is therefore uniformly continuous.

(c) The function h is uniformly continuous. One way to see this is to use the mean value theorem. Note that, since $x \ge 1$,
$$|h'(x)| = \frac{1}{x^2} \le 1$$
so by the mean value theorem, for all $x, y \in [1, \infty)$ we have
$$|h(y) - h(x)| \le |y - x|.$$
Now let $\varepsilon > 0$. Taking $\delta = \varepsilon$, we see that for $x, y \in [1, \infty)$, if $|x-y| < \delta$ then $|h(x) - h(y)| < \varepsilon$, so h is uniformly continuous.

134. Let $s_n(x) = \sum_{i=1}^n f_i(x)$. Then (s_n) converges uniformly to f, and by Proposition 8.1.6, this is equivalent to $\sup\{|s_n(x) - f(x)|\} \to 0$ as $n \to \infty$. So

$$\begin{aligned}\sup\{|f_n(x)|\} &= \sup\{|s_n(x) - s_{n-1}(x)|\} \\ &\le \sup\{|s_n(x) - f(x)| + |f(x) - s_{n-1}(x)|\} \\ &\le \sup\{|s_n(x) - f(x)|\} + \sup\{|f(x) - s_{n-1}(x)|\} \\ &\quad \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

So by Proposition 8.1.6, (f_n) converges uniformly to 0.

135. (a) Let $M_n = 1/n^2$. Note that

$$\frac{1}{n^2 + x^2} \le \frac{1}{n^2}$$

for all $x \in \mathbb{R}$, and the series $\sum_{n=1}^\infty M_n$ converges. So by the Weierstrass M-test, the series $\sum_{n=1}^\infty \frac{1}{n^2+x^2}$ converges uniformly on $\mathbb{R}$.
It also converges uniformly on $[0, 1]$ by the same argument.

(b) Observe that, for each $n \in \mathbb{N}$,

$$\sup\left\{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| \mid x \in \mathbb{R}\right\} = \infty$$

so the series cannot converge uniformly on $\mathbb{R}$. (The series converging uniformly means the sequence of partial sums s_n converging uniformly. But then you would have $s_{n+1} - s_n$ converging uniformly to the zero function, which is not the case.)
But let

$$M_n = \sup\left\{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| \mid x \in [0, 1]\right\} = \frac{1}{(2n+1)!}.$$

Then the terms of the series are bounded above by M_n, and $\sum_{n=0}^\infty M_n$ converges. So the series converges uniformly on $[0, 1]$.

(c) $\sin(nx)$ does not converge to 0 as $n \to \infty$ at certain values of x, (for example $x = \pi/4 \in [0, 1]$), so the series does not converge, let alone uniformly, either on $[0, 1]$ or $\mathbb{R}$.

(d) (*) The series $\sum_{n=1}^\infty \frac{\sin(nx)}{n}$ does not converge uniformly on $[0, 1]$. So it does not converge uniformly on $\mathbb{R}$ either. To see this, suppose it does converge uniformly. Then the limit $f(x)$ would be a continuous function on $[0, 1]$. Since f is continuous at 0, we have

$$0 = f(0) = \lim_{N\to\infty} f(\frac{\pi}{N}) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sin(n\pi/N)}{n}.$$

On the other hand,

$$\sum_{n=1}^{N} \frac{\sin(n\pi/N)}{n} \geq \frac{1}{N}\left(\sum_{n=1}^{N} \sin(n\pi/N)\right)$$
$$= \frac{1}{N}\left(\frac{\sin(\frac{N}{2}\frac{\pi}{N})\sin(\frac{N+1}{2}\frac{\pi}{N})}{\sin(\frac{\pi}{2N})}\right).$$

As $N \to \infty$, this has limit

$$\lim_{N\to\infty} \frac{1}{N}\left(\frac{\sin(\frac{(1+1/N)\pi}{2})}{\sin(\frac{\pi}{2N})}\right) = \lim_{x\to 0} \frac{2}{\pi}\frac{x}{\sin(x)} = \frac{2}{\pi},$$

giving a contradiction.

136. (a) Let $M_n = a^n$. Then for $x \in [0,a]$, we have $|x^n| \leq M_n$, and since $|a| < 1$, the series $\sum_{n=1}^{\infty} M_n$ converges.
Hence, by the Weierstrass M-test, the series converges uniformly for $x \in [0,a]$.

(b) Let
$$S_n(x) = 1 + x + x^2 + \cdots + x^n.$$
Let
$$A_n = \sup\{S_n(x) \mid x \in [0,1)\} = S_n(1) = 1 + 1 + \cdots + 1 = n.$$
The sequence (A_n) does not converge. It follows that the sequence of partial sums $(S_n(x))$ does not converge uniformly on $[0,1)$. Hence the series does not converge uniformly on $[0,1)$.

137. Let $f_n : [0,2\pi] \to \mathbb{R}$ be given by

$$f_n(x) = \frac{\sin(nx)}{n^2}.$$

Then f_n is continuous. Note that

$$|f_n(x)| \leq \frac{1}{n^2}$$

for all $x \in [0,2\pi]$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Hence, by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly for $x \in [0,2\pi]$. The uniform limit of a series of continuous functions is continuous, so f is therefore continuous.

Chapter 9 Problems: Applications

138. (a) Let

$$a_n = \frac{n^2}{2n+1}$$

Observe that

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n^2/(2n+1)}{(n+1)^2/(2n+3)} = \frac{n^2(2n+3)}{(n+1)^2(2n+1)}$$
$$= \left(\frac{1}{1+1/n}\right)^2 \frac{2+3/n}{2+1/n}.$$

This converges to 1 as $n \to \infty$.
So the series has radius of convergence 1.

(b) This is a geometric series, with ratio $-2x$. So it converges absolutely when $|-2x| < 1$, and does not converge when $|-2x| > 1$, that is to say it converges absolutely when $|x| < 1/2$, and does not converge when $|x| > 1/2$. Thus we have radius of convergence $1/2$.

(c) This is a geometric series, with ratio x^2. So it converges absolutely when $|x^2| < 1$, and does not converge when $|x^2| > 1$, that is to say it converges absolutely when $|x| < 1$, and does not converge when $|x| > 1$. Thus we have radius of convergence 1.

(d) Let

$$c_n = \frac{(-1)^n x^{2n}}{(2n)!}$$

Then

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{|x|^2}{(2n+1)(2n+2)} \to 0$$

as $n \to \infty$. So by the ratio test, the power series always converges absolutely, and we have radius of convergence ∞.

(e) Let $y = x^3$, and consider the power series

$$\sum_{n=0}^{\infty} n^{3n} y^n$$

This series has radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{n^{3n}}{(n+1)^{3n+3}}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3n}\frac{1}{(n+1)^3}$$
$$= \lim_{n\to\infty}\left(\frac{1}{(1+1/n)^n}\right)^3 \frac{1}{(n+1)^3} = \frac{1}{e^3}.0 = 0,$$

So it converges only when $y = 0$.
Since $y = x^3$, the original series converges only when $x = 0$, that is to say it has radius of convergence 0.

139. Let $|x| < R$. Then the series $\sum_{n=0}^{\infty} a_n t^n$ converges uniformly for $t \in [-x, x]$. Hence, by the uniform convergence theorem for integrals

$$\int_0^x \sum_{n=0}^{\infty} a_n t^n \, dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n \, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

140. Let $0 < r < 1$ and let x be such that $|x| \leq r$. Set $M_n = r^n$. Then the series $\sum_{n=0}^{\infty} M_n$ converges as $|r| < 1$.

Since $x^n \leq r^n$, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} x^n$ converges uniformly.

As a geometric series, the limit is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Using the uniform convergence theorem for integrals,

$$\int_{-r}^{r} \frac{1}{1-x} \, dx = \int_{-r}^{r} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \int_{-r}^{r} x^n \, dx.$$

For n odd, we have

$$\int_{-r}^{r} x^n \, dx = 0.$$

For n even,

$$\int_{-r}^{r} x^n \, dx = 2 \int_0^r x^n \, dx = \frac{2r^{n+1}}{n+1}.$$

So,

$$\int_{-r}^{r} \frac{1}{1-x} \, dx = 2\left(r + \frac{r^3}{3} + \frac{r^5}{5} + \cdots\right).$$

On the other hand,

$$\int_{-r}^{r} \frac{1}{1-x} \, dx = -\log(1-r) + \log(1+r) = \log\left(\frac{1+r}{1-r}\right).$$

Putting it all together, the result follows.

141. (a) When $|x| < 1$, the geometric series

$$1 - x^2 + x^4 - x^6 + \cdots$$

converges, to

$$\frac{1}{1+x^2}.$$

That is, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

for $|x| < 1$. and the power series on the right has radius of convergence 1.

(b) Integrating the above term by term, which is valid within the radius of convergence,

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

(c) Since

$$\arctan(1) = \frac{\pi}{4},$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

142. The series

$$\sum_{n=0}^{\infty} n!x^n$$

has radius of convergence 0.

143. We work by contradiction, so suppose that e is rational. Since e is positive, we then have $e = \frac{a}{b}$ where $a, b \in \mathbb{N}$. So $be = a$, and so $b!e \in \mathbb{N}$.

But

$$b!e = b! + \frac{b!}{1!} + \frac{b!}{2!} + \cdots + \frac{b!}{b!} + \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots$$

Notice that the terms up to $b!/b!$ are all integers. Let

$$\begin{aligned} R &= \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots \\ &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \end{aligned}$$

Notice that

$$R < \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots$$

By the formula for the sum of a geometric series,

$$0 < R < \frac{1}{b+1} \times \frac{1}{1 - \frac{1}{b+1}} = \frac{1}{b}.$$

In particular, R is not a natural number. Therefore $b!e$ is not a natural number. But this is a contradiction. So e is irrational.