MAS221(2015-16) Exam Solutions

1. (i) (a) Given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n > N, |a_n - l| < \epsilon$. **(2)**

(b) Given any two positive real numbers x, y there exists $n \in \mathbb{N}$ so that $nx > y$. **(1)**

(c) Guess limit is 1/5, then given any $\epsilon > 0$, we must find $N \in \mathbb{N}$ so that $n > N \Rightarrow |1/5 - 7/\sqrt{n} - 1/5| = 7/\sqrt{n} < \epsilon$. **(1)**
Now $7/\sqrt{n} < \epsilon$ if and only if $n > 49/\epsilon^2$ **(1)**.
Using the Archimedean property, (with $x = 1, y = 49/\epsilon^2$), there exists $N \in \mathbb{N}$ so that $N > 49/\epsilon^2$. **(1)**
Then for all $n > N$ we have $7/\sqrt{n} < 7/\sqrt{N} < \epsilon$, as required. **(1)**

(ii) (a) Proof by induction. True for $n = 1$, assume true for some n, then $9 \leq x_n^2 \leq 16$, and so

$$\frac{1}{7}(9 + 12) \leq \frac{1}{7}(x_n^2 + 12) \leq \frac{1}{7}(16 + 12),$$

i.e. $3 \leq x_{n+1} \leq 4$, as required. **(2)** Hence, by induction, the required bound holds for all $n \in \mathbb{N}$. **(1)**

(b) For all $n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{7}(x_n^2 + 12) - x_n \quad \textbf{(1)} \\ &= \frac{1}{7}(x_n^2 - 7x_n + 12) \\ &= \frac{1}{7}(x_n - 3)(x_n - 4) \quad \textbf{(1)} \end{aligned}$$

But by (a), $(x_n - 3) \geq 0$ and $(x_n - 4) \leq 0$, and so $(x_n - 3)(x_n - 4) \leq 0$. **(1)** Hence $x_{n+1} \leq x_n$, as required **(1)**.

(c) Since the sequence is monotonic decreasing and bounded below, it converges to a limit $l = \inf_{n \in \mathbb{N}} x_n$ **(1)**
By algebra of limits in (b) (or alternatively, can use (a) and solve quadratic)

$$0 = \lim_{n \to \infty} x_{n+1} - \lim_{n \to \infty} x_n = \frac{1}{7}(\lim_{n \to \infty} x_n - 3)(\lim_{n \to \infty} x_n - 4),$$

and hence $l = 3$ or $l = 4$. **(2)** But $x_1 = 3.5$ and sequence is monotonic decreasing, so we must have $l = 3$. **(1)**

(iii) (a) A sequence (a_n) is Cauchy if given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $m, n > N$ we have $|a_m - a_n| < \epsilon$. **(1)**

(b) Choose $\epsilon = 1$ (say). Then for any $n > N$, by (a) and the triangle inequality

$$|a_n| \leq |a_{N+1}| + |a_n - a_{N+1}| \leq 1 + |a_{N+1}|, \quad (\mathbf{2})$$

so the required bound is

$$K = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a_{N+1}|\}. \quad (\mathbf{1})$$

(c) If (a_n) is Cauchy, then by (b) it is bounded, so there exists $K > 0$ so that $|a_n| \leq K$ for all $n \in \mathbb{N}$. **(1)**
If (b_n) is null, then given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n > N$, $|b_n| < \epsilon/K$. **(1)**
Then for the same ϵ and N if $n > N$, we have

$$|a_n b_n| \leq K|b_n| < \epsilon,$$

as required. **(1)**

2. (i) Using the triangle inequality,

$$\begin{aligned} |a| &= |a + b - b| \quad (\mathbf{1}) \\ &\leq |a| + |a - b| \quad (\mathbf{1}) \end{aligned}$$

and so $|a| - |b| \leq |a - b|$ **(1)**
Now interchange a and b to get $|b| - |a| \leq |b - a| = |a - b|$.
The result follows since $||a| - |b|| = \max\{|a| - |b|, |b| - |a|\}$. **(1)**

(ii) Given any sequence (x_n) converging to a with $x_n \in D_f \setminus \{a\}$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} f(x_n) = f(a)$. **(1)**[1] f continuous at a if $a \in D_f$ and $\lim_{x\to a} f(x) = f(a)$. **(1)**

(iii) (a) For all $x \in D_f \setminus \{a\}$ we have by (i)

$$0 < ||f(x)| - |l|| \leq |f(x) - l|, \quad (\mathbf{1})$$

and the result follows by the sandwich rule (or can use $\epsilon - \delta$ criterion). **(1)**

[1] This is the definition given in the course, which was later shown to be equivalent to the $\epsilon - \delta$ criterion. Any student correctly quoting the latter as the definition will also receive the full mark.

(b) If f is continuous at $a, l = f(a)$ and so $|l| = |f(a)|$. The result follows. **(1)**

(c) $f(x) = x$. **(1)**

(iv) (a) Since $-1 \leq \cos(y) \leq 1$ for all $y \in \mathbb{R}$, we have $-|x| \leq x\cos(1/x) \leq |x|$ for all $x \neq 0$. **(1)** Then by the sandwich rule, $\lim_{x\to 0} x\cos(1/x) = 0$ **(1)**
and the required continuous extension of f is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases} \quad (\mathbf{1})$$

(b) By product and chain rules, if $x \neq 0$,

$$\tilde{f}'(x) = f'(x) = \cos(1/x) + \frac{1}{x}\sin(1/x), \quad (\mathbf{1})$$

$\tilde{f}$ is not differentiable at zero. **(1)** Indeed for $x \neq 0$,

$$\frac{\tilde{f}(x) - \tilde{f}(0)}{x} = \cos(1/x), \quad (\mathbf{1})$$

which has no limit at $x = 0$. To see this first consider a sequence (x_n) for which $x_n = 1/2n\pi$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \cos(1/x_n) = 1$. **(1)** Now consider a sequence (y_n) for which $y_n = 1/(\pi/2 + 2n\pi)$ for all $n \in \mathbb{N}$. Then again $\lim_{n\to\infty} y_n = 0$, but this time $\lim_{n\to\infty} \cos(1/y_n) = 0$. **(1)**

(v) (a) Rolle's Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$. **(1)**

(b) By algebra of limits, g is continuous on $[a, b]$ and differentiable on (a, b). **(1)** We have $g(a) = f(a)$ and $g(b) = f(b) - \frac{f(b)-f(a)}{b-a}.(b - a) = f(a) = g(a)$. **(1)** Hence by Rolle's theorem, there exists $c \in (a, b)$ so that $g'(c) = 0$. **(1)** But then $f'(c) = \alpha$ and the result follows. **(1)**

(c) Let $a \leq x < y \leq b$ be arbitrary. By the mean value theorem, there exists $c \in (x, y)$ so that

$$f(y) - f(x) = f'(c)(y - x) < 0,$$

and so f is strictly monotonic decreasing. **(2)**

3. (i) (a) $B(\boldsymbol{a}, r) = \{\boldsymbol{x} = \mathbb{R}^k \mid |\boldsymbol{x} - \boldsymbol{a}| < r\}$. **(1)**
We call $U \subseteq \mathbb{R}^k$ *open* if for all $\boldsymbol{a} \in U$ there exists $r > 0$ such that $B(\boldsymbol{a}, r) \subseteq U$. **(1)**

(b) We call $f\colon \mathbb{R}^k \to \mathbb{R}^l$ continuous if for each $\boldsymbol{a} \in \mathbb{R}^k$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\boldsymbol{x} - \boldsymbol{a}| < \delta$ implies $|f(\boldsymbol{x}) - f(\boldsymbol{a})| < \varepsilon$. **(1)**

(c) We define the inverse image $f^{-1}[U] = \{\boldsymbol{x} \in U$ such that $f(\boldsymbol{x}) \in U\}$. **(1)**
Let $\boldsymbol{a} \in f^{-1}[U]$. Then $f(\boldsymbol{a}) \in U$, so we have $\varepsilon > 0$ where $B(f(\boldsymbol{a}), \varepsilon) \subseteq U$. **(1)**
Since f is continuous, we have $\delta > 0$ such that if $|\boldsymbol{x} - \boldsymbol{a}| < \delta$, then $|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| < \varepsilon$. **(1)**
Hence, if $\boldsymbol{x} \in B(\boldsymbol{a}, \delta)$, then $f(\boldsymbol{x}) \in B(f(\boldsymbol{a}, \varepsilon) \subseteq U$, so $\boldsymbol{x} \in f^{-1}[U]$. Thus $f^{-1}[U]$ is also open. **(2)**

(ii) The interval (a, ∞) is open. The others listed are not open. **(1 each)**

(iii) (a) Define a continuous map $f\colon \mathbb{R}^2 \to \mathbb{R}$ by the formula $f(x, y) = x^2 + y^2$. Then $A = f^{-1}[(-\infty, 1)]$ which is open since the interval $(-\infty, 1)$ is open. **(1) answer, (1) justification**

(b) As in (a), $B = f^{-1}[(1, 2)]$, which is open as $(1, 2)$ is open. **(1) answer, (1) justification**

(c) Let $(x, y) \in C$. Then $x \leq 1$. Observe $(1, 0) \in C$. For any $r > 0$ the ball $B((1, 0), r)$ contains the element $(1 + r/2, 0)$, which is an element (x, y) where $x > 1$. So the ball $B((1, 0), r)$ contains elements not in C. It follows that C is not open. **(1) answer, (1) justification**

4. (i) (a) The sequence (f_n) *converges pointwise* to f if for each $t \in [a, b]$, we have

$$\lim_{n \to \infty} f_n(t) = f(t).$$

(1)
The sequence (f_n) *converges uniformly* to f if for for all $\varepsilon > 0$, we have $N \in \mathbb{N}$ such that $|f_n(t) - f(t)| < \varepsilon$ for all $n \geq N$ and $t \in [a, b]$. **(1)**

(b) Let $t_0 \in [a, b]$. We want to prove f is continuous at t_0. Let $\varepsilon > 0$. Then:

- We have $N \in \mathbb{N}$ such that $|f_n(t) - f(t)| < \frac{\varepsilon}{3}$ whenever $n \geq \mathbb{N}$, $t \in [a, b]$. (uniform convergence)

- We have $\delta > 0$ such that $|f_N(t) - f_N(t_0)| < \frac{\varepsilon}{3}$ whenever $|t - t_0| < \delta$. (f_N is continuous)

So, let $|t-t_0| < \delta$. Then we want to show that $|f(t)-f(t_0)| < \varepsilon$ follows, and we have shown that f is continuous at the (arbitary, chosen at the start) point t_0 as required. **(3)**
Using the above two conditions, if $|t - t_0| < \delta$, then:

$$|f(t)-f(t_0)| \leq |f(t)-f_N(t)|+|f_N(t)-f_N(t_0)|+|f_N(t_0)-f_(t_0)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

(2)
Since f_n and f are continuous, they are integrable. **(1)**
Let $\varepsilon > 0$. Then we have $N \in \mathbb{N}$ such that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2(b-a)}$$

whenever $n \geq N$, for all $t \in [a, b]$. **(2)**
Hence by linearity and standard inequalities for integrals

$$\left| \int_a^b f_n(t)\ dt - \int_a^b f(t)\ dt \right| \leq \int_a^b |f_n(t)-f(t)|\ dt \leq \frac{(b-a)\varepsilon}{2(b-a)} = \frac{\varepsilon}{2} < \varepsilon$$

whenever $n \geq N$. **(2)**
The result now follows.

(c) The sequence $f_n(t)$ converges pointwise to the function

$$f(t) = \begin{cases} 0 & t < 1 \\ 1 & t = 1 \end{cases}$$

which is not continuous, so the convergence is not uniform by the above. **(1) sequence of functions, (1) idea of using continuity, (1) correct details**

(ii) (a) Let $f_n\colon [a, b] \to \mathbb{R}$ be a sequence of functions. Suppose we have a sequence of real numbers (M_n) such that $|f_n(x)| \leq M_n$ for all n, and all $x \in [a, b]$. such that the sum $\sum_{n=1}^{\infty} M_n$ converges. Then the sequence $\sum_{n=1}^{\infty} f_n$ converges uniformly (and absolutely). **(2)**

(b) Let $|x| \leq r$. Set $M_n = r^n$. Then the series $\sum_{n=0}^{\infty} M_n$ converges as $|r| < 1$. **(1)**
Since $x^n \leq r^n$, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} x^n$ converges uniformly. **(1)**

As a geometric series, the limit is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(1)

(c) By uniform convergence of the series, and the result for uniform convergence and integrals: **(1)**

$$\int_{-r}^{r} \frac{1}{1-x}\, dx = \sum_{n=0}^{\infty} \int_{-r}^{r} x^n\, dx$$

(1)

For n odd, we have **(1)**

$$\int_{-r}^{r} x^n\, dx = 0$$

For n even:

$$\int_{-r}^{r} x^n\, dx = 2\int_{0}^{r} x^n\, dx = \frac{2r^{n+1}}{n+1}$$

(1)

Finally.

$$\int_{-r}^{r} \frac{1}{1-x}\, dx = -\log(1-r) + \log(1+r) = \log\left(\frac{1-r}{1+r}\right)$$

(1)

Putting it all together, the result follows.

(iii) (a) Let $f\colon I \to \mathbb{R}$. We say f is *uniformly continuous* on I if for all $\varepsilon > 0$ we have $\delta > 0$ such that for all $x, y \in [a, b]$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. **(2)**

(b) The function g is continuous, with a closed bounded domain, and is therefore uniformly continuous. **(2)**

(c) The function h is uniformly continuous. One way to see this is to use the mean value theorem. Note that, since $x \geq 1$,

$$|h'(x)| = \frac{1}{x^2} \leq 1$$

so by the mean value theorem, for all $x, y \in [1, \infty)$ we have

$$|(h(y) - h(x)| < |y - x|$$

Hence let $\varepsilon > 0$. Taking $\delta = \varepsilon$, we see that for $x, y \in [1, \infty)$, if $|x-y| < \delta$ then $|h(x) - h(y)| < \varepsilon$, so h is uniformly continuous. **(3)**