MAS221(2016-17) Exam Solutions

1. (i) A is (a) *bounded above* if there exists $K \in \mathbb{R}$ so that $a \leq K$ for all $a \in A$ (**1**); (b) it is *bounded below* if there exists $L \in \mathbb{R}$ so that $a \geq L$ for all $a \in A$ (**1**).

   e.g. the set $\{-n; n \in \mathbb{N}\}$ is bounded above (by zero) but not below. (**1**)

   (ii) Suppose that for some $\epsilon > 0$ such an a cannot be found, then for all $a \in A, a \leq \sup(A) - \epsilon$. (**1**) But then $\sup(A) - \epsilon$ is a smaller upper bound for A than $\sup(A)$, and we have a contradiction. (**2**)

   (iii) The sequence (a_n) is *monotonic increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. (**1**)

   (iv) Since (a_n) is bounded above, α exists by the completeness property. (**1**) By (ii) given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $a_N > \alpha - \epsilon$. (**1**) But (a_n) is monotonic increasing, so for all $n > N$ we have $a_n \geq a_{N+1} \geq a_N > \alpha - \epsilon$. (**1**) Since $a_n \leq \alpha$,

$$|\alpha - a_n| = \alpha - a_n < \epsilon, \text{ for all } n \geq N,$$

   and the result follows. (**2**)

   (v) (a) It's clearly true for $n = 1$. (**1**) Assume true for some value of n. Then

$$a_{n+1} \geq \frac{57.0 + 1}{1 + 57} = \frac{1}{58} > 0, \ (\mathbf{1})$$

$$\begin{aligned} a_{n+1} - 1 &= \frac{57a_n + 1}{a_n + 57} - 1 \\ &= \frac{56(a_n - 1)}{a_n + 57} \ (\mathbf{1}) \\ &\leq \frac{56(1 - 1)}{57} = 0, \ (\mathbf{1}) \end{aligned}$$

   so $0 \leq a_{n+1} \leq 1$, and the required result is true, by induction. (**1**)

   (b) For all $n \in \mathbb{N}$,

$$\begin{aligned} a_{n+1} - a_n &= \frac{57a_n + 1}{a_n + 57} - a_n \\ &= \frac{57a_n + 1 - a_n^2 - 57a_n}{a_n + 57} \ (\mathbf{1}) \\ &= \frac{1 - a_n^2}{a_n + 57} \geq 0 \text{ (by (a))}, \ (\mathbf{1}) \end{aligned}$$

and so $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, i.e. (a_n) is monotonic increasing. (**1**)

(c) By (a) the sequence is bounded above (by 1), and by (b) it is monotonic increasing. So it converges to a limit by (iv). (**1**) Let $\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. (**1**) Now take limits on both sides of the general formula, and use algebra of limits to get

$$\alpha = \frac{57\alpha + 1}{\alpha + 57}. \ (\mathbf{1})$$

From this, we get $\alpha^2 = 1$, i.e. $\alpha = \pm 1$. but $a_1 = 0$ and the sequence is monotonic increasing. So we must have $\lim_{n\to\infty} a_n = 1$. (**2**)

2. (i) (a) f is bounded if there exists $K \geq 0$ so that $|f(x)| \leq K$ for all $x \in [a, b]$, OR f is both bounded above and below. (**2**)

(b) The set $A := \{f(x), x \in [a, b]\}$ is non-empty, and bounded both above and below. (**1**) So both the sup and the inf exist by the completeness axiom. (**1**)

(c) There exist $c, d \in [a, b]$ with $f(c) = \sup_{x\in[a,b]} f(x)$ and $f(d) = \inf_{x\in[a,b]} f(x)$. (**2**)

(d) f must be continuous. (**1**)

(e) No, e.g. $f(x) = 1/x$ on $(0, 1)$. (**2**)

(ii) (a) The limit does not exist. (**1**). To see this consider e.g the sequences (x_n) and (y_n) where $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{\pi/2+2n\pi}$ for all $n \in \mathbb{N}$. Both converge to zero. (**1**) But $\lim_{n\to\infty} g(x_n) = 1$ while $\lim_{n\to\infty} g(y_n) = 0$. (**2**)

(b) $x \to \sin(1/x)$ is continuous as it is the composition of two continuous functions. (**1**) Then f is continuous by algebra of limits as it is the product of two continuous functions. (**1**)

(c) Since $-1 \leq \sin(1/x) \leq 1$, we have $-x^2 \leq x^2 \sin(1/x) \leq x^2$ for all $x \neq 0$. (**2**) But $\lim_{x\to 0} x^2 = 0$, so by the sandwich rule, $\lim_{x\to 0} f(x) = 0$. (**2**) Then the required continuous extension is given by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}. \ (\mathbf{1})$$

(d) For $x \neq 0$, by product and chain rules of differentiation

$$f_1'(x) = 2x \sin(1/x) - \cos(1/x) \quad (\mathbf{2})$$

When $x = 0$,

$$\begin{aligned} f_1'(0) &= \lim_{h\to 0} \frac{f_1(h) - f_1(0)}{h} \\ &= \lim_{h\to 0} h\sin(1/h) = 0, \end{aligned}$$

by a similar argument to the result of (c). (**2**)
But f_1' is not continuous at $x = 0$ as $\lim_{x\to 0} f_1'(x)$ does not exist, by (a). (**1**)

3. (i) (a)

$$L\int_a^b f(t)\,dt = \sup\left\{\int_a^b s(t)\,dt \mid s \text{ is a step function, } s(t) \le f(t) \text{ for all } t\right\}. \quad (\mathbf{1})$$

$$U\int_a^b f(t)\,dt = \inf\left\{\int_a^b s(t)\,dt \mid s \text{ is a step function, } s(t) \ge f(t) \text{ for all } t\right\}. \quad (\mathbf{1})$$

$f : [a, b] \to \mathbb{R}$ is *Riemann integrable* if

$$L\int_a^b f(t)\,dt = U\int_a^b f(t)\,dt. \quad (\mathbf{1})$$

(b)

$$\int_0^4 r(t)\,dt = 1 + e + e^8 + e^{27}, \quad (\mathbf{2})$$
$$\int_0^4 s(t)\,dt = e + e^8 + e^{27} + e^{64}. \quad (\mathbf{2})$$

(c) The function f is continuous, so it is Riemann integrable on a bounded interval. (**1**)

(d) The function r is a step function with $r(t) \le e^{t^3}$ for all $t \in [0, 4]$, since the two functions have common values at $t = 0, 1, 2, 3$ and e^{t^3} is increasing. (**1**)
Hence

$$1 + e + e^8 + e^{27} = \int_0^4 r(t)\,dt \le L\int_0^4 e^{t^3}\,dt = \int_0^4 e^{t^3}\,dt. \quad (\mathbf{1})$$

Similarly, s is a step function with $s(t) \ge e^{t^3}$ for all $t \in [0, 4]$ (**1**) and

$$\int_0^4 e^{t^3}\,dt = U\int_0^4 e^{t^3}\,dt \le \int_0^4 s(t)\,dt = e + e^8 + e^{27} + e^{64}. \quad (\mathbf{1})$$

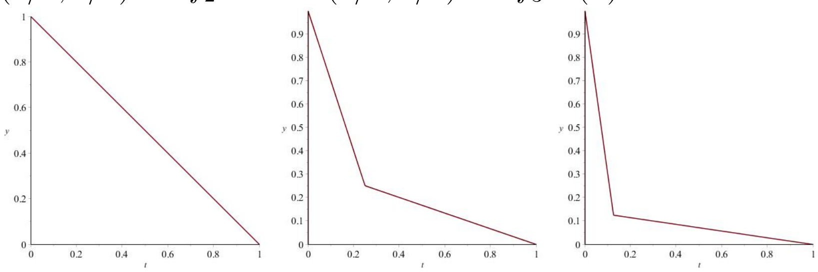
(ii) (a) The sequence (f_n) *converges pointwise* to a function $f\colon [a,b] \to \mathbb{R}$ if for each $t \in [a,b]$, we have

$$\lim_{n\to\infty} f_n(t) = f(t). \quad (\mathbf{1})$$

[This form is fine. Or they might write out the ϵ, N version of the limit.]

(b) The sequence (f_n) *converges uniformly* to a function $f\colon [a,b] \to \mathbb{R}$ if for all $\varepsilon > 0$, we have $N \in \mathbb{N}$ such that $|f_n(t) - f(t)| < \varepsilon$ for all $n \geq N$ and all $t \in [a,b]$. (**1**)

(iii) (a) The graphs of f_1, f_2, f_3 are below. The slope changes at $(1/4, 1/4)$ for f_2 and at $(1/8, 1/8)$ for f_3. (**3**)



(b) The sequence (f_n) converges pointwise to the function $f : [0,1] \to \mathbb{R}$ with

$$f(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t \neq 0. \quad (\mathbf{1}) \end{cases}$$

To see this, firstly, $f_n(0) = 1$ for all n, so $f_n(0) \to 1$. (**1**)
Now suppose $t > 0$. Then there is some N such that $\frac{1}{2^N} < t$. So for $n > N$, $f_n(t) = \frac{1-t}{2^n-1} \to 0$ as $n \to \infty$. (**2**)

(c) No. If the sequence converged uniformly this would be to the same limit function f. (**1**)
For each n, the function f_n is continuous. This is clear on $[0, \frac{1}{2^n})$ and on $(\frac{1}{2^n}, 1]$, since f_n is a linear function there.
So it's sufficient to check continuity at $t = \frac{1}{2^n}$. For this,

$$\lim_{t \nearrow \frac{1}{2^n}} f_n(t) = 1 - (2^n - 1)\frac{1}{2^n} = \frac{1}{2^n} = f(\frac{1}{2^n}). \quad (\mathbf{1})$$

A result from the course says that the uniform limit of a sequence of continuous functions is continuous. (**1**)
But f is not continuous at 0, since for any sequence (x_n) converging to 0 with each $x_n > 0$, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq f(0) = 1$. (**1**)

4. (i) (a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(2n)!((n+1)!)^2}{(n!)^2(2(n+1))!} = \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+1)(2n+2)}$$
$$= \lim_{n\to\infty}\frac{(1+1/n)(1+1/n)}{(2+1/n)(2+2/n)} = \frac{1}{4} < 1. \qquad (\mathbf{2})$$

So by the ratio test, the series converges. (**1**)

(b)

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 4. \qquad (\mathbf{2})$$

(ii) Let $f_n : [0, 2\pi] \to \mathbb{R}$ be given by $f(t) = \frac{\cos(nt)}{n^2}$, a continuous function. (**1**)

Note that $|f_n(t)| = \left|\frac{\cos(nt)}{n^2}\right| \le \frac{1}{n^2}$ for all $t \in [0, 2\pi]$ (**1**)

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (**1**)

Hence, by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} f_n(t)$ converges uniformly and we can define a function by

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2} \qquad (\mathbf{1})$$

The uniform limit of a sequence of continuous functions is continuous, so this function is continuous. (**1**)

(iii) We have $|\mathbf{a}_n| = \sqrt{x_n^2 + y_n^2} \le C$, so $|x_n| \le C$ and $|y_n| \le C$ for all n. (**1**)

Thus (x_n) is a bounded sequence of real numbers and so it has a convergent subsequence, say (x_{n_k}). (**1**)

Consider the subsequence (y_{n_k}) of (y_n). This is again a bounded sequence of real numbers so it has a convergent subsequence $(y_{n_{k_l}})$. (**1**)

The sequence $(x_{n_{k_l}})$ is a subsequence of the convergent sequence (x_{n_k}), so it converges. (**1**)

Thus the subsequence $(\mathbf{a}_{n_{k_l}}) = ((x_{n_{k_l}}, y_{n_{k_l}}))$ converges in each component, so it converges. (**1**)

(iv) (a) The *open ball* with radius r around $\mathbf{x}$ is the set

$$B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^k \mid |\mathbf{x} - \mathbf{y}| < r\}. \quad (\mathbf{1})$$

(b) We call a subset $U \subseteq \mathbb{R}^k$ an *open set* if for all $\mathbf{x} \in U$ there is some $\delta > 0$ such that $B(\mathbf{x}, \delta) \subseteq U$. (**1**)

(c) Let $\mathbf{x} \in U \cap V$. Then $\mathbf{x} \in U$ and $\mathbf{x} \in V$.
As $\mathbf{x} \in U$ and U is open, we have $\delta_1 > 0$ such that $B(\mathbf{x}, \delta_1) \subseteq U$.
As $\mathbf{x} \in V$ and V is open, we have $\delta_2 > 0$ such that $B(\mathbf{x}, \delta_2) \subseteq V$. (**2**)
Let $\delta = \min(\delta_1, \delta_2)$. Then

$$B(\mathbf{x}, \delta) \subseteq B(\mathbf{x}, \delta_1) \subseteq U, \qquad B(\mathbf{x}, \delta) \subseteq B(\mathbf{x}, \delta_2) \subseteq V,$$

so $B(\mathbf{x}, \delta) \subseteq U \cap V$, and $U \cap V$ is open. (**2**)

(v) (a) No, this is not open in $\mathbb{R}^2$. For example, any open ball around the point $(1, 0)$ contains points which are not in the set. (**1**)

(b) Yes, this is an open set in $\mathbb{R}^2$.
Let p_1, p_2 be the projection maps $\mathbb{R}^2 \to \mathbb{R}$ onto the first and second coordinates respectively. We can write the set as $U_1 \cap U_2$, where $U_1 = p_1^{-1}((0, \infty))$ and $U_2 = p_2^{-1}((0, \infty))$. Since the projections are continuous, the pre-images U_1 and U_2 of an open interval in $\mathbb{R}$ are open. And the intersection of two open sets is open. (**3**)

[Alternatively, students may show openness directly from the definition.]