MAS221 Analysis 2017-18 – Solutions IV

Solutions for Chapter 6 : Series

104. Let
$$s_n = a_1 + \cdots + a_n.$$
Since the series $\sum_{n=1}^{\infty} a_n$ converges, we have $\lim_{n\to\infty} s_n = L$, for some $L \in \mathbb{R}$.

As (s_{n+1}) is a subsequence of (s_n), it follows that $\lim_{n\to\infty} s_{n+1} = L$. Now $a_{n+1} = s_{n+1} - s_n$ and using the algebra of limits,
$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} s_{n+1} - s_n = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = L - L = 0.$$
Therefore the sequence (a_{n+1}), and thus also the sequence (a_n), converges with limit 0.

105. (a) Observe that
$$\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \ln(1+1) + \ln(1+\frac{1}{2}) + \cdots + \ln(1+\frac{1}{n})$$
$$= \ln\left((1+1)(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n})\right).$$
Now putting everything over a common denominator
$$(1+1)(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n}) = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\cdots\cdot\frac{n+1}{n} = n+1.$$
So
$$\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \ln(n+1)$$

(b) Certainly $\ln(n+1)$ does not converge as $n \to \infty$. So the series does not converge.

106. (a) We calculate
$$\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} = \frac{4}{(2k)(k+1)(2(k+2))} = a_k.$$

(b) Using part (a),

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+2} - \sum_{k=1}^{n}\frac{1}{k+1} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} + \sum_{k=2}^{n-1}\frac{1}{k+1} - \sum_{k=1}^{n}\frac{1}{k+1} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} + \sum_{k=2}^{n-1}\frac{1}{k+1} - \frac{1}{2} - \frac{1}{n+1} - \sum_{k=2}^{n-1}\frac{1}{k+1} \\ &= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}.\end{aligned}$$

(c) Letting $n \to \infty$, we see that the series converges with

$$\sum_{k=1}^{\infty} a_k = \frac{1}{4}.$$

107. (a) Since we have assumed $a_n \neq 0$ for all n, we can consider the sequence $(\frac{b_n}{a_n})$. Then

$$\lim_{n\to\infty}\frac{b_n}{a_n} = \frac{1}{\lim_{n\to\infty}\frac{a_n}{b_n}} = 1,$$

using the algebra of limits.

(b) Since $\lim_{n\to\infty}\frac{b_n}{a_n} = 1$, taking $\epsilon = 1$, there is some N such that for $n > N$, $|\frac{b_n}{a_n} - 1| < 1$. Thus for $n > N$, $|b_n - a_n| < |a_n|$, and using the triangle inequality, $|b_n| = |b_n - a_n + a_n| \leq |b_n - a_n| + |a_n| < 2|a_n|$.

(c) By assumption $\sum_{n=1}^{\infty} a_n$ converges absolutely, that is, $\sum_{n=1}^{\infty} |a_n|$ converges. Therefore, so does $\sum_{n=1}^{\infty} 2|a_n|$.

Hence, by the comparison test, the series

$$\sum_{n=1}^{\infty} |b_n|$$

converges, that is $\sum_{n=1}^{\infty} b_n$ converges absolutely.

108. (a) Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

So we have

$$\sum_{n=1}^{N}\frac{1}{n(n+1)} = \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+1} = 1 - \frac{1}{N+1}.$$

If we let $N \to \infty$, we see

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

In particular, the series converges.

(b) Let $a_n = \frac{1}{n(n+1)}$ and let $b_n = \frac{1}{n^2}$. All the terms a_n are positive, so we have seen in part (a) that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Observe that

$$\lim_{n\to\infty} \frac{1/n(n+1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

So by question 107, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(c) Observe that, for $\alpha \geq 2$, we have $n^\alpha \geq n^2$, and so

$$\frac{1}{n^\alpha} \leq \frac{1}{n^2}.$$

So the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges by the comparison test.

109. (a) Let

$$a_n = \frac{3^n + 5^n}{4^n} = \left(\frac{3}{4}\right)^n + \left(\frac{5}{4}\right)^n$$

Note that $a_n \to \infty$ as $n \to \infty$. So the series $\sum_{n=1}^{\infty} a_n$ does not converge.

(b) We know that $1/n \to 0$ as $n \to \infty$, and that cos is a continuous function, with $\cos 0 = 1$, so

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1.$$

It follows that the series

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

does not converge.

(c) Let

$$\begin{aligned} a_n &= \frac{\sqrt{n^2+1} - \sqrt{n^2-1}}{n} = \frac{(n^2+1)-(n^2-1)}{n(\sqrt{n^2+1}+\sqrt{n^2-1})} \\ &= \frac{2}{n(\sqrt{n^2+1}+\sqrt{n^2-1})}. \end{aligned}$$

Now

$$\sqrt{n^2+1} \geq \sqrt{n^2} = n, \qquad \sqrt{n^2-1} \geq 0,$$

so

$$0 \leq a_n \leq \frac{2}{n^2}.$$

The series $\sum_{n=1}^{\infty} 1/n^2$ converges, so, by the comparison test, so does the series $\sum_{n=1}^{\infty} a_n$.

(d) Let

$$a_n = \frac{(n!)^2}{(2n)!}$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{(2n)!((n+1)!)^2}{(2n+2)!(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{(1+1/n)^2}{(2+1/n)(2+2/n)}$$

so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4} < 1.$$

Hence by the ratio test the series $\sum_{n=0}^{\infty} a_n$ converges.

110. The n-th partial sum is $s_n = 1 + 2r + 3r^2 + \cdots + nr^{n-1}$. Then

$$(1-r)s_n = 1 + r + r^2 + \cdots + r^{n-1} - nr^n = \frac{1-r^n}{1-r} - nr^n.$$

Thus

$$s_n = \frac{1-r^n}{(1-r)^2} - \frac{nr^n}{1-r}.$$

Since $|r| < 1$, $\lim_{n\to\infty} r^n = 0$. So $\lim_{n\to\infty} s_n = \frac{1}{(1-r)^2}$, as required.

111. Following the hint, we let $a = \tan^{-1}\left(\frac{1}{2n-1}\right)$ and $b = \tan^{-1}\left(\frac{1}{2n+1}\right)$ and use the identity

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}.$$

Then

$$\tan(a-b) = \frac{\frac{1}{2n-1} - \frac{1}{2n+1}}{1 + \frac{1}{(2n-1)(2n+1)}} = \frac{1}{2n^2}.$$

Thus

$$\tan^{-1}\left(\frac{1}{2n^2}\right) = a - b = \tan^{-1}\left(\frac{1}{2n-1}\right) - \tan^{-1}\left(\frac{1}{2n+1}\right).$$

So

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k-1}\right) - \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k+1}\right) \\ &= \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2n+1}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2n+1}\right). \end{aligned}$$

Since $\tan^{-1}$ is continuous at 0, $\lim_{n\to\infty} \tan^{-1}\left(\frac{1}{2n+1}\right) = \tan^{-1} 0 = 0$. So $\lim_{n\to\infty} s_n = \frac{\pi}{4}$, as required.

112. (a) False. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, but $\lim_{n\to\infty} \frac{1}{n} = 0$.

(b) True. The partial sums converge to 0, so $\sum_{n=1}^{\infty} a_n$ converges to 0, by definition.

(c) True. As the terms a_n are positive, the sequence of partial sums (s_n) is monotonic increasing. Since it is also bounded above, it converges, so $\sum_{n=1}^{\infty} a_n$ converges.

(d) True. Since $\sum_{n=1}^{\infty} |a_n|$ converges, we have $\lim_{n\to\infty} |a_n| = 0$. In particular, there is some N such that for all $n > N$, $|a_n| < 1$. Then, for $n > N$,

$$a_n^2 = |a_n||a_n| < |a_n|.$$

It follows by the comparison test that $\sum_{n=1}^{\infty} a_n^2$ converges.