**MAS221 Analysis 2017-18 – Solutions to Exercises III**

*Solutions to Chapter 5 problems*

80. (a) This is Definition 5.2.1 in the notes: we say that $f$ is *differentiable* at $a \in D_f$ if $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists (and is finite). Or, equivalently, $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists (and is finite).

    (b) There exists $f'(a) \in \mathbb{R}$, so that given any sequence $(h_n)$ that converges to 0, with $h_n \neq 0$, we have $\lim\limits_{n\to\infty} \dfrac{f(a+h_n)-f(a)}{h_n} = f'(a)$.

    (c) Given $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |h| < \delta$, then $\left|\dfrac{f(a+h)-f(a)}{h} - f'(a)\right| < \epsilon$.

81. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) \\ &= -\frac{1}{x(x+h)} \to -\frac{1}{x^2}, \text{ as } h \to 0. \end{aligned}$$

Thus $f'(x) = -\frac{1}{x^2}$ for all $x \in \mathbb{R} \setminus \{0\}$.
The extended function is not continuous at $x = 0$, since $\lim_{x\downarrow 0} \frac{1}{x} = \infty$. Therefore it is not differentiable at $x = 0$, by Theorem 5.2.4.

82. As in the hint, let $g(h) = e^{kh} - 1 - kh$. Then

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{1}{h}(e^{k(x+h)} - e^{kx}) \\ &= e^{kx}\frac{1}{h}(e^{kh}-1) = e^{kx}\left(k + \frac{g(h)}{h}\right) \to ke^{kx}, \text{ as } h \to 0, \end{aligned}$$

using the fact that $\lim_{h\to 0} g(h)/h = 0$. Thus $f'(x) = ke^{kx}$.

83. Using the product and chain rules for differentiation, and that sin is differentiable, if $x \neq 0$, $f$ is differentiable with $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$. But $\frac{f(x)-f(0)}{x} = \sin(1/x)$ has no limit as $x \to 0$ (see Problem 50), so $f$ is not differentiable at 0.

84. Again using the product and chain rules and standard derivatives, for $x \neq 0$, $f$ is differentiable with $f'(x) = 2x\sin(1/x) - \cos(1/x)$. At $x = 0$,

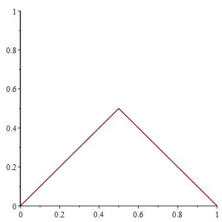
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin(1/x) = 0,$$

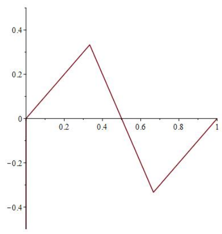
by Problem 51. So $f$ is differentiable at 0 with $f'(0) = 0$. For the second derivative, with $x \neq 0$, we have

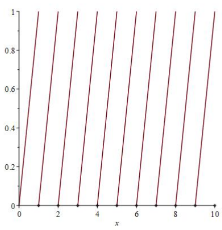
$$f''(x) = 2\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right).$$

But $f''$ doesn't exist at $x = 0$, as in Problem 83.

85. We give sketches of simple examples of functions as described. There should be a "corner" at the relevant points.

(a) 


(b) 


86. It helps to sketch the graph. 


$f$ is differentiable for all $x \in \mathbb{R} \setminus \mathbb{Z}$. For such points, taking $h$ sufficiently small, we have $[x+h] = [x]$ and so $\frac{(x+h)-[x+h]-x+[x]}{h} = \frac{h}{h}$. Thus $f'(x) = 1$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$. If $x \in \mathbb{Z}$, then $f$ is not continuous at $x$ and so cannot be differentiable there.

87. First observe that if $f$ is differentiable at $a$, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a)-f(a-h)}{h}.$$

So

$$\begin{aligned} \frac{f(a+h)-f(a-h)}{2h} &= \frac{1}{2}\left(\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right) \\ &\to \frac{1}{2}2f'(a) = f'(a), \text{ as } h \to 0. \end{aligned}$$

On the other hand, $f(x) = |x|$ is not differentiable at $x = 0$. But

$$\lim_{h\downarrow 0} \frac{|h| - |-h|}{2h} = 0.$$

88. Define $g_a : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$g_a(h) = \frac{f(a+h)-f(a)}{h}.$$

First suppose that $f$ is differentiable at $a$, then $f'_+(a) = \lim_{h\downarrow 0} g_a(h) = f'(a)$ and $f'_-(a) = \lim_{h\uparrow 0} g_a(h) = f'(a)$. Conversely, suppose that $f'_+(a) = f'_+(a) = l$ (say). Then $\lim_{h\downarrow 0} g_a(h) = \lim_{h\uparrow 0} g_a(h)$ and so $\lim_{h\to 0} g_a(h)$ exists by Theorem 3.3.2, and equals $l$. But then $f$ is differentiable and $l = f'(a)$.

89. (a) True: $f$ is continuous at zero as $f(0) = 0 = \lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x)$.

(b) True: $f'(0)$ exists and is zero. To see this compute

$$f'_+(0) = \lim_{h\downarrow 0} \frac{h^2}{h} = \lim_{h\downarrow 0} h = 0.$$

and

$$f'_-(0) = \lim_{h\uparrow 0} \frac{-h^2}{h} = \lim_{h\uparrow 0} -h = 0.$$

(c) True: $f'$ is continuous at zero since $f' : \mathbb{R} \to \mathbb{R}$ is given by

$$f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 2x & \text{if } x > 0 \end{cases},$$

so $f'(0) = \lim_{x\uparrow 0} f'(x) = \lim_{x\downarrow 0} f'(x) = 0.$

(d) False: $f''_+(0) = \lim_{h\downarrow 0} \frac{2h-0}{h} = 2$, $f''_-(0) = \lim_{h\uparrow 0} \frac{-2h-0}{h} = -2$, and so $f''(0)$ does not exist.

90. (a) Yes: $f$ is differentiable on $[a,b]$ and hence continuous on $[a,b]$ by Theorem 5.2.4, so it attains its sup and inf on $[a,b]$ by Theorem 4.3.4, and these are the maximum and minimum (respectively).

(b) No: the maximum or minimum could be $f(a) = f(b)$. If $f(a)$ is not the maximum value, then this must occur in $(a,b)$. Similarly for the minimum. If $f(a) = f(b)$ is both the maximum and minimum value, then $f$ is constant, and the value occurs in $(a,b)$.

91. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^n$. Then $g$ is differentiable on $[0,1]$, with $g(0) = g(1) = 0$. So by Rolle's theorem, there exists $c \in (0,1)$ such that $g'(c) = 0$, i.e.$f(c) = 0$.

92. (a) Apply the mean value theorem to any interval $[x,y]$ with $a \leq x < y \leq b$, to find there exists $c \in (x,y)$ for which

$$f(y) - f(x) = f'(c)(y - x) = 0.$$

Thus $f(x) = f(y)$ for all $x, y \in [a,b]$, i.e. $f$ is constant on $[a,b]$.

(b) Define $f = h - g$, then the function $f$ satisfies all the conditions of (a), and the result is immediate.

93. By the mean value theorem, $f(b) = f(a) + f'(c)(b-a)$ for some $c \in (a,b)$. So, since $m \leq f'(c) \leq M$, for all $c \in (a,b)$,

$$f(a) + m(b-a) \leq f(b) \leq f(a) + M(b-a).$$

94. The function $f$ is differentiable on $[0,\pi]$ with $f'(x) = -\sin(x) < 0$ on $(0,\pi)$. So by Corollary 5.5.2 it is strictly decreasing on $[0,\pi]$. Hence by Theorem 4.3.7 it has a strictly decreasing inverse on $[f(\pi), f(0)] = [-1,1]$ which is continuous on $(-1,1)$. This is precisely the function $f^{-1}(x) = \cos^{-1}(x)$ or $\arccos(x)$. Also $x \mapsto -\sin(x)$ is continuous on $(0,\pi)$, so by Theorem 5.5.3, $f^{-1}$ is differentiable on $(-1,1)$, and

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(x)} = -\frac{1}{\sin(x)} = -\frac{1}{\sqrt{1-\cos^2(x)}} \\ &= -\frac{1}{\sqrt{1-y^2}},\end{aligned}$$

for all $y \in (-1,1)$, where $y = f(x) = \cos(x)$.

95. The polynomial $p$ is of odd degree so it has at least one real root by Corollary 4.3.3. Also $p'(x) = 3x^2 + r > 0$, for all $x \in \mathbb{R}$, so $p$ is strictly monotonic increasing on any closed interval $[a, b]$, and hence on the whole of $\mathbb{R}$, by Corollary 5.5.2. Then by Theorem 4.3.7, $p$ is invertible and hence injective, and so there is exactly one zero.

96. Apply the mean value theorem to the function $f(x) = x^p$ on the interval $[x, 1]$. Then there exists $c \in (x, 1)$ such that

$$\frac{f(x) - f(1)}{x - 1} = f'(c).$$

Thus

$$1 - x^p = pc^{p-1}(1 - x) < p(1 - x).$$

97. Consider the first case. Here we have $f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} = \lim_{h \to 0} \frac{f'(a+h)}{h} < 0$. In particular, $f''_+(a) = \lim_{h \downarrow 0} \frac{f'(a+h)}{h} < 0$, and so $f'(a + h) < 0$ for sufficiently small $h$, (say $0 < h < \delta$), in which case, by Corollary 5.5.2, $f$ is strictly decreasing on $[a, a + \delta]$. We also have $f''_-(a) = \lim_{h \uparrow 0} \frac{f'(a+h)}{h} < 0$, and so $f'(a + h) > 0$ for sufficiently small $h$, (say $-\delta_1 < h < 0$), in which case, by Corollary 5.5.2, $f$ is strictly increasing on $[a - \delta_1, a]$. Then it follows that $f$ has a local minimum at $a$. The other case is similar.

98. The function $h$ from the hint is continuous on $[a, b]$ and differentiable on $(a, b)$. After some algebraic manipulation, we obtain

$$h(a) = h(b) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)},$$

and so by Rolle's theorem, there exists $c \in (a, b)$ so that $h'(c) = 0$ and so $f'(c) = \rho g'(c)$, and the result follows.

99. (a) $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \cos(x) = 1$,
$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = \lim_{x \to 0} -\sin(x) = 0$.

(b) Using a standard trig. identity,

$$\begin{aligned} & \frac{\sin(x + h) - \sin(x)}{h} - \cos(x) \\ = & \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} - \cos(x) \\ = & \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h} - 1\right) \\ \to & \ 0 \text{ as } h \to 0, \end{aligned}$$

where we have used (a) and the algebra of limits.

(c) By (a), the required continuous extension is given by:

$$\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

It is differentiable at zero, as for all $h \neq 0$,

$$\frac{\tilde{f}(h) - \tilde{f}(0)}{h} = \frac{\sin(h)}{h^2} - \frac{1}{h}.$$

Now by using Maclaurin's theorem, we have $\sin(h) = h - \frac{h^3}{6}\cos(\theta h)$, for some $\theta$ with $0 < \theta < 1$. So $\frac{\tilde{f}(h)-\tilde{f}(0)}{h} = -\frac{h}{6}\cos(\theta h) \to 0$ as $h \to 0$.

100. Applying l'Hôpital's rule to the given expression yields

$$\lim_{h \downarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h \downarrow 0} \frac{f'(a+h) - f'(a-h)}{2h},$$

and then the result follows from the result of Problem 87.

101. (a) Since the left limits diverge to infinity at $a$ it follows that given any $R > 0$, there exists $K \in (a, b)$ so that $f(x) > R$ and $g(x) > R$ for all $a < x < K$. Just choose $R = 1$ (say) and we are done with the first part. The second part is an application of Cauchy's mean value theorem, followed by a little algebra.

(b) Multiply both sides of the result of (a) by $\frac{1-g(K)/g(x)}{1-f(K)/f(x)}$. The rest is algebra.

(c) Just take limits as $x \downarrow a$ of both sides in (b).

(d) We have $c = K + \theta(K - a)$, where $0 \leq \theta \leq 1$. Then

$$0 \leq |c - a| = |K - a|.|1 + \theta| \leq 2|K - a|,$$

so by the sandwich rule, $c \to a$ as $K \to a$, and the result follows by taking limits as $K \to a$ on both sides of the result of (c).

102. (a) $\frac{x^n}{n!}e^{\theta x}$, (b) $(-1)^n \frac{x^{2n}}{(2n)!}\sin(\theta x)$, (c) $(-1)^{n+1}\frac{x^{2n}}{(2n)!}\sin(\theta x)$, where in all three cases $0 < \theta < 1$.

103. Using Maclaurin's theorem, we can find $0 < \theta < 1$ and $0 < \phi < 1$, so that for all $x \in \mathbb{R}$,

$\cos(x) = 1 - \frac{x^2}{2}\cos(\theta x) \geq 1 - \frac{x^2}{2}$, since $\cos(\theta x) \leq 1$, and

$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}\cos(\phi x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, since $\cos(\phi x) \leq 1$.