MAS221 Analysis 2017-18 – Exercises IV

Problems for Chapter 6 : Series

104. Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$. (This is Proposition 6.1.3 in the notes; the proof was left as an exercise.)

105. (a) Show that

$$\sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) = \ln(n+1).$$

(b) Deduce that the series

$$\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k}\right).$$

does not converge.

106. Let

$$a_k = \frac{1}{k(k+1)(k+2)}$$

(a) Show that

$$a_k = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$$

(b) Deduce an expression for the sum

$$\sum_{k=1}^{n} a_k.$$

(c) Hence prove that the series $\sum_{k=1}^{\infty} a_k$ converges and find the limit.

107. Let (a_n) be a sequence of non-zero real numbers and assume that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Let (b_n) be a sequence of non-zero real numbers. Suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

(a) Show that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 1.$$

(b) Deduce that there is some N such that $|b_n| \le 2|a_n|$ for all $n > N$.

(c) Hence show that the series $\sum_{n=1}^{\infty} b_n$ converges absolutely.

108. (a) By writing $\frac{1}{n(n+1)}$ as a partial fraction, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges.

(b) By using the result in question 107, prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(c) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges whenever $\alpha \geq 2$.

109. Using any of the tests for convergence, or other reasoning, determine which of the following series converge. Justify your answers.

(a)

$$\sum_{n=1}^{\infty} \frac{3^n + 5^n}{4^n}$$

(b)

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

(c)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1} - \sqrt{n^2-1}}{n}$$

Hint: show that $\frac{\sqrt{n^2+1}-\sqrt{n^2-1}}{n} \leq \frac{2}{n^2}$.

(d)

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

110. Show that, if $|r| < 1$, the series $\sum_{n=1}^{\infty} nr^{n-1}$ converges to $\frac{1}{(1-r)^2}$.

Hint: Show that

$$(1-r)s_n = \frac{1-r^n}{1-r} - nr^n,$$

where s_n is the n-th partial sum of the series.

111. Show that

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{2n^2}\right) = \frac{\pi}{4}.$$

Hint: Let $a = \tan^{-1}\left(\frac{1}{2n-1}\right)$ and $b = \tan^{-1}\left(\frac{1}{2n+1}\right)$ and use the identity

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}.$$

112. For each of the following statements, say whether it is true or false, giving reasons for your answers.

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(c) If $a_n > 0$ for all n and the sequence of partial sums $(s_n) = (a_1 + \cdots + a_n)$ is bounded, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) If the series $\sum_{n=1}^{\infty} |a_n|$ is convergent, then so is $\sum_{n=1}^{\infty} a_n^2$.