MAS221 Analysis 2017-18 – Exercises III

Problems for Chapter 5: Differentiation

(*Problems labelled * may be more demanding.*)

A version of the Chapter 5 problems already appeared at the end of Exercises II. These problems have been modified and updated a little, so please use this version.

80. (a) Give the definition of what it means for a real-valued function f to be differentiable at $a \in D_f$.

    (b) State carefully what this means in terms of limits of sequences.

    (c) State carefully what this means in terms of the $\epsilon - \delta$ criterion.

81. Give a rigorous proof that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$ is differentiable for all $x \in \mathbb{R} \setminus \{0\}$ from the definition of the derivative and find $f'(x)$ explicitly. Can we extend the function so that it is differentiable on the whole of $\mathbb{R}$ by defining its value at zero to be zero?

82. Let $k \in \mathbb{R} \setminus \{0\}$. Give a rigorous proof from the definition of the derivative that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{kx}$ is differentiable for all $x \in \mathbb{R}$, and find $f'(x)$ explicitly. You may use the fact that $e^{kh} - 1 - kh = g(h)$, where $\lim_{h\to 0} g(h)/h = 0$, (which follows from the series expansion).

83. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

    is differentiable at every $x \neq 0$ but fails to be differentiable at $x = 0$. [For $x \neq 0$, you can use standard facts about derivatives. At $x = 0$, you need to use the definition of the derivative as a limit.]

84. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

    is differentiable at every $x \in \mathbb{R}$. What can you say about its second derivative?

85. (a) Sketch the graph of any continuous function $f : [0,1] \to \mathbb{R}$ which is not differentiable at $x = 1/2$. [You do *not* need to give a formula.]

    (b) Sketch the graph of any continuous function $f : [0,1] \to \mathbb{R}$ which is not differentiable at $x = 1/3$ and is not differentiable at $x = 2/3$. [You do *not* need to give a formula.]

86. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x - [x]$ for all $x \in \mathbb{R}$. (Recall that if $x \in \mathbb{R}$, $[x]$ is the integer part of x, i.e. $[x] = \max\{n \in \mathbb{Z} \,|\, n \leq x\}$.) Explain carefully at which points is differentiable, and find the value of its derivative there. [It helps to sketch the graph!]

87. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a then

$$\lim_{h \downarrow 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a).$$

By considering $f(x) = |x|$, show that the limit on the left-hand side may exist, even when f is not differentiable at a.

88. Prove Theorem 5.2.7, i.e. show that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a if and only if it is both right and left differentiable there, and these two derivatives both agree there, in which case $f'(a)$ is their common value.

89. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$

Determine whether each of the following is true or false.

    (a) f is continuous at 0.
    (b) $f'(0)$ exists.
    (c) f' is continuous at 0.
    (d) $f''(0)$ exists.

90. (a) Must any differentiable function $f : [a,b] \to \mathbb{R}$ have a maximum and minimum value? Why?

    (b) If $f : [a,b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, must f have a maximum and minimum value in (a,b)?

91. Let $a_0, a_1, \ldots, a_n$ be real numbers such that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Consider $f : \mathbb{R} \to \mathbb{R}$ give by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Show that there is some $c \in (0,1)$ such that $f(c) = 0$. [Hint: Integrate the function f term–by–term, and think about how to use Rolle's theorem.]

92. Use the mean value theorem to show the following.

    (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $f'(c) = 0$ for all $c \in (a,b)$, then f is constant on $[a,b]$.

    (b) If g and h are both continuous on $[a,b]$ and differentiable on (a,b) with $h'(x) = g'(x)$ for all $x \in (a,b)$. then there exists $k \in \mathbb{R}$ so that $h(x) = g(x) + k$, for all $x \in [a,b]$.

93. If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and there exist $m, M \in \mathbb{R}$ such that $m \leq f'(c) \leq M$, for all $c \in (a,b)$, show that

$$f(a) + m(b-a) \leq f(b) \leq f(a) + M(b-a).$$

94. Show that the restriction of $f(x) = \cos(x)$ to $[0,\pi]$ has an inverse defined on $[-1,1]$, which is differentiable on $(-1,1)$.

95. If $r > 0$ and $q \in \mathbb{R}$ show that the polynomial $p(x) = x^3 + rx + q$ has exactly one real zero.

96. By applying the mean value theorem to the function $f(x) = x^p$ on $[x,1]$, show that if $p > 1$ and $x \in (0,1)$ then

$$1 - x^p < p(1-x).$$

97. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at a with $f'(a) = 0$. If $f''(a) < 0$, show that f has a local maximum at a, while if $f''(a) > 0$, show that f has a local minimum at a.

98. Prove Cauchy's mean value theorem (Theorem 5.5.4 in the notes.) [Hint: Define $h(x) = f(x) - \rho g(x)$, where $\rho = \frac{f(b)-f(a)}{g(b)-g(a)}$ and show how to obtain the result by applying Rolle's Theorem to h.]

99. (a) Use l'Hôpital's rule to find $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 0} \frac{\cos(x)-1}{x}$.

(b) Use the result of (a) to give a rigorous proof from the definition of the derivative that $f(x) = \sin(x)$ is differentiable at every $x \in \mathbb{R}$, with $f'(x) = \cos(x)$.

(c) Consider the function $f(x) = \sin(x)/x$ with $D_f = \mathbb{R} \setminus \{0\}$. Find a continuous extension of f to the whole of $\mathbb{R}$. Is your extended function differentiable at $x = 0$?

100. Use l'Hôpital's rule to show that if f is twice differentiable at a, then

$$f''(a) = \lim_{h \downarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

101. * The purpose of this question is to prove the following version of l'Hôpital's rule: Suppose that f and g are each differentiable on (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \downarrow a} f(x) = \lim_{x \downarrow a} g(x) = \infty$, then

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \lim_{x \downarrow a} \frac{f'(x)}{g'(x)},$$

whenever the limit on the right hand side is finite.

(a) First show there exists $K \in (a, b)$ so that $f(x) \neq 0, g(x) \neq 0$ for all $x \in (a, K)$ and that there exists $c \in (a, K)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(K)}{g(x) - g(K)} = \frac{f(x)}{g(x)} \left(\frac{1 - f(K)/f(x)}{1 - g(K)/g(x)} \right).$$

(b) Deduce that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \left[1 + \frac{f(K)/f(x) - g(K)/g(x)}{1 - f(K)/f(x)} \right].$$

(c) Show that $\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$.

(d) Finally, show that you can take limits as $K \to a$ to deduce the result.

102. Use Taylor (or Maclaurin's) theorem to find the remainder terms appearing in the following expansions (where $x \in \mathbb{R}$):

(a) $e^x = 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x)$.

(b) $\sin(x) = x - \frac{x^3}{3!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x)$.

(c) $\cos(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n+1}(x)$.

103. Prove that for all $x \in \mathbb{R}$:

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$