MAS221 Analysis, Semester 2, Exercises V
Chapters 7, 8 and 9

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(Exercises labelled * may be more demanding.)

Chapter 7 Problems: Integration

113. (a) Define step functions $r, s\colon \mathbb{R} \to \mathbb{R}$ by

$$r(t) = \chi_{[0,1)} + e\chi_{[1,2)} + e^4\chi_{[2,3]}, \qquad s(t) = e\chi_{[0,1]} + e^4\chi_{(1,2]} + e^9\chi_{(2,3]}.$$

Evaluate the integrals

$$\int_0^3 r(t)\,dt, \qquad \int_0^3 s(t)\,dt.$$

(b) Why is the function $f\colon [0,3] \to \mathbb{R}$ defined by $f(x) = e^{x^2}$ Riemann integrable?

(c) Prove that

$$1 + e + e^4 \leq \int_0^3 e^{x^2}\,dx \leq e + e^4 + e^9.$$

[Hint: there's no need to calculate the integral in the middle; use the previous parts to prove the inequalities.]

114. Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$.

(a) Let $n \in \mathbb{N}$. Let $r_n : [0,1] \to \mathbb{R}$,

$$r_n(t) = 0\chi_{[0,1/n)} + \left(\frac{1}{n}\right)^2 \chi_{[1/n,2/n)} + \cdots + \left(\frac{n-1}{n}\right)^2 \chi_{[(n-1)/n,1)}.$$

Show that $r_n(t) \leq f(t)$ for all $t \in [0,1]$, and calculate the integral $\int_0^1 r_n(t)\,dt$, of the step function r_n. You may use without proof the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

(b) Let $n \in \mathbb{N}$. Let $s_n : [0,1] \to \mathbb{R}$,

$$s_n(t) = \left(\frac{1}{n}\right)^2 \chi_{(0,1/n]} + \left(\frac{2}{n}\right)^2 \chi_{(1/n,2/n]} + \cdots + 1\chi_{((n-1)/n,1]}.$$

Show that $s_n(t) \geq f(t)$ for all $t \in [0,1]$, and calculate the integral $\int_0^1 s_n(t)\,dt$, of the step function s_n.

(c) Use the above to calculate the integral

$$\int_0^1 x^2\ dx.$$

[Of course, you know how to calculate this integral more easily, but the point here is to do it from first principles, using the definition of the Riemann integral.]

115. Which of the following functions are Riemann integrable? Justify your answers.

(a) The function $f\colon [1,2] \to \mathbb{R}$ defined by

$$f(x) = \frac{\exp(\sin x)}{x^3 + 5}.$$

(b) The function $k\colon [0,1] \to \mathbb{R}$ defined by

$$k(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} & \frac{1}{2} \leq x < \frac{3}{4}, \\ \frac{3}{4} & \frac{3}{4} \leq x < \frac{7}{8}, \\ \frac{7}{8} & \frac{7}{8} \leq x < \frac{15}{16}, \\ \vdots & \vdots \end{cases}$$

116. Let $r, s\colon [a,b] \to \mathbb{R}$ be step functions.

(a) Show that the product $rs\colon \mathbb{R} \to \mathbb{R}$, defined by the formula $(rs)(t) = r(t)s(t)$, is a step function.

(b) Prove that

$$\left(\int_a^b r(t)s(t)\ dt\right)^2 \leq \left(\int_a^b r(t)^2\ dt\right)\left(\int_a^b s(t)^2\ dt\right).$$

[Hint: start from $(\alpha r(t) + \beta s(t))^2 \geq 0$ and let

$$\alpha = \left(\int_a^b s(t)^2\,dt\right)^{\frac{1}{2}}, \qquad \beta = \left(\int_a^b r(t)^2\,dt\right)^{\frac{1}{2}}. \qquad]$$

(c) Let $f, g\colon [a,b] \to \mathbb{R}$ be step functions. Define

$$\|f\| = \left(\int_a^b f(t)^2 \, dt \right)^{\frac{1}{2}}.$$

Prove that $\|f+g\| \le \|f\| + \|g\|$.

117. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $a, b\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions. Prove that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(y)\, dy = b'(x) f(b(x)) - a'(x) f(a(x)).$$

118. Define a function $L\colon (0,\infty) \to \mathbb{R}$ by

$$L(x) = \int_1^x \frac{1}{t} \, dt.$$

Prove the following directly from the definition of L via an integral (that is, *without* using any properties of the function ln).

(a) L is differentiable, and $L'(x) = \frac{1}{x}$.

(b) $L(xy) = L(x) + L(y)$ for all $x, y > 0$.

(c) $L(x/y) = L(x) - L(y)$ for all $x, y > 0$.

119. (*) Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuous function satisfying the formula $f(xy) = f(x) + f(y)$ for all $x, y > 0$.

(a) Prove that $f(x^a) = af(x)$ for all $x > 0$ and $a \in \mathbb{R}$.
[Hint: Prove this first for $\mathbb{N}$, then $\mathbb{Z}$, then $\mathbb{Q}$, and finally extend to $\mathbb{R}$ by continuity.]

(b) Prove that $f(x) = f(e)\ln(x)$ for all $x > 0$.

(c) For the above function L, prove that $L(x) = \ln x$.

120. (*) Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Prove that there is a number $x \in [a,b]$ such that

$$\int_a^x f(t)\, dt = \int_x^b f(t)\, dt.$$

[Hint: apply the intermediate value theorem to the function F given by

$$F(x) = \int_a^x f(t)\, dt - \int_x^b f(t)\, dt. \qquad]$$

121. (a) Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Suppose there are $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Prove that there is a number $\mu \in [m, M]$ such that

$$\int_a^b f(x)\ dx = (b-a)\mu.$$

(b) Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Prove that there is some $\xi \in [a,b]$ such that

$$\int_a^b f(x)\ dx = (b-a)f(\xi).$$

122. Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and let $g\colon [a,b] \to [0,\infty)$ be integrable. Prove that there is some $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(\xi)\int_a^b g(x)\ dx.$$

Do we need the assumption $g(x) \geq 0$? Justify your answer.

Chapter 8 Problems: Sequences and series of functions

126. Consider the sequence of functions (f_n), where $f_n : [0,\pi] \to \mathbb{R}$ is defined by $f_n(x) = \sin^n(x)$. Show that the sequence (f_n) converges pointwise. Does the sequence (f_n) converge uniformly? Justify your answer.

127. For each of the following sequences of functions (f_n) determine the pointwise limit (if it exists) on the indicated interval, and decide whether (f_n) converges uniformly to this limit.

(a) $f_n(x) = x^{1/n}$, $x \in [0,1]$.

(b)
$$f_n(x) = \begin{cases} 0 & x \leq n, \\ x-n & x \geq n, \end{cases} \qquad x \in \mathbb{R}.$$

(c) $f_n(x) = e^x/x^n$, $x \in (1,\infty)$.

(d) $f_n(x) = e^{-nx^2}$, $x \in [-1,1]$.

(e) $f_n(x) = e^{-x^2}n$, $x \in \mathbb{R}$.

128. For each of the following sequences of functions (g_n) find the pointwise limit, and determine whether the sequence converges uniformly on $[0,1]$, and on $[0,\infty)$.

(a) $g_n(x) = x/n$.

(b) $g_n(x) = x^n/(1+x^n)$.

(c) $g_n(x) = x^n/(n+x^n)$.

129. For each of the following sequences of functions (h_n), where $h_n\colon [0,1] \to \mathbb{R}$, find the pointwise limit, if it exists, and in that case determine whether the sequence converges uniformly.

(a) $h_n(x) = (1 - x/n)^2$.

(b) $h_n(x) = x - x^n$.

(c) $h_n(x) = \sum_{k=0}^{n} x^k$.

130. Define $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{n + \cos x}{2n + \sin^2 x}.$$

(a) Find the pointwise limit of the sequence of functions (f_n).

(b) Show that the sequence (f_n) converges uniformly.

(c) Calculate

$$\lim_{n\to\infty} \int_2^7 f_n(x)\, dx.$$

131. (a) Let $n \in \mathbb{N}$. Show that we can define a continuous function $f_n\colon [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & x = 0, \\ \frac{x^{1/n}-1}{\ln x} & 0 < x < 1, \\ \frac{1}{n} & x = 1. \end{cases}$$

(Note: you only need check continuity at $x = 0$ and $x = 1$.)

(b) Does the sequence (f_n) converge uniformly to a limit? Justify your answer. If you wish, you may assume without proof that each function f_n is monotone increasing.

(c) Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

132. Compute the limits

$$\lim_{n\to\infty} \int_0^1 \frac{e^{t^4}}{n}\, dt, \qquad \lim_{n\to\infty} \int_1^2 t^{2-((\sin nt)/n)}\, dt,$$

justifying any procedures you use.

133. Which of the following functions are uniformly continuous? Justify your answers.

(a) The function $f\colon (0,2] \to \mathbb{R}$ defined by $f(x) = 1/x$.

(b) The function $g\colon [1,2] \to \mathbb{R}$ defined by $g(x) = 1/x$.

(c) The function $h\colon [1,\infty) \to \mathbb{R}$ defined by $h(x) = 1/x$.

134. Let $f_n : [a,b] \to \mathbb{R}$. Suppose that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f. Show that the sequence (f_n) converges uniformly to the zero function.

135. By using the Weierstrass M-test or otherwise, for each of the following series, determine whether it converges uniformly on $\mathbb{R}$ and whether it converges uniformly on $[0,1]$.

(a)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$$

(b)
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(c)
$$\sum_{n=1}^{\infty} \sin(nx)$$

(d) (*)
$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

136. (a) Show the series $\sum_{n=1}^{\infty} x^n$ converges uniformly for $x \in [0,a]$ whenever $0 < a < 1$.

(b) Does the series converge uniformly on $[0,1)$? Explain.

137. Prove that there is a function $f\colon [0,2\pi] \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

and that this function is continuous.

Chapter 9 Problems: Applications

138. Find the radius of each of convergence of the following power series.

(a)
$$\sum_{n=0}^{\infty} \frac{n^2}{2n+1} x^n$$

(b)

$$1 - 2x + 4x^2 - 8x^3 + 16x^4 - \cdots$$

(c)

$$1 + x^2 + x^4 + x^6 + x^8 + \cdots$$

(d)

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

(e)

$$\sum_{n=0}^{\infty} (nx)^{3n}$$

139. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Is it true that

$$\int_0^x \sum_{n=0}^{\infty} a_n t^n \, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

whenever $|x| < R$?

Justify your answer using any theorems you need from the course.

140. By integrating the power series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

between $-r$ and r, show that

$$\log\left(\frac{1+r}{1-r}\right) = 2\left(r + \frac{r^3}{3} + \frac{r^5}{5} + \cdots\right)$$

whenever $0 < r < 1$. Justify each step.

141. (a) Express the function

$$\frac{1}{1+x^2}$$

as a power series. What is the radius of convergence?

(b) Use part (a) and question 139 to find a power series converging to the function $\arctan(x)$.

(c) Use part (b) to find a series converging to $\frac{\pi}{4}$.

142. Write down a series with radius of convergence 0.

143. Use the power series for $\exp(x)$ to prove that the number e is irrational. [Hint: suppose that $e = \exp(1)$ is rational, equal to $\frac{a}{b}$, for natural numbers a, b, and obtain a contradiction by using the power series expression to show that $b!e$ is not a natural number.]